纯粹数学与应用数学专著 第 25 号

# 强极限定理

林正炎 陆传荣 著

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## 内 容 简 介

本书深入地研究了 Wiener 过程的几乎处处极限性质并把某些结果推广于一类 Gauss 过程中，介绍了独立但不必同分布的随机变量序列部分和的增量的几乎处处极限结果，研究了由无穷维 Ornstein-Uhlenbeck 过程生成的过程的样本性质.

本书可供大学概率统计专业学生、研究生和从事概率统计工作的人员阅读.



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**强 极 限 定 理**

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《纯粹数学与应用数学专著》丛书

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# 序　言

近20年来强逼近和强收敛性理论已经成为概率论学科中一个十分活跃的研究领域。1981 年 M.Csörgő 和 P. Révész 的专著“Strong Approximations in Probability and Statistics”总结了直至 70 年代末为止有关的基本结果。在该书的引言中作者指出了若干可能进一步研究的方向，例如多参数过程、高维欧氏空间（或Banach空间）中的过程、不同分布且（或）不独立的随机变量序列等。

近年来，该书中的许多结果已经被很大程度地改进或推广。Wiener 过程的 a.s.（轨道）极限性质一直被不少作者广泛深入地加以研究。随机变量序列部分和的增量的a.s.极限结果已被林正炎和邵启满强化并推广到了独立但不必同分布的情形。某些与 Wiener 过程有关的 Gauss 过程的轨道性质也已被深入地加以研究，本书的目的就在于总结上述工作。

我们非常感激M.Csörgő教授（加拿大 Carleton大学），没有他的鼓励和支持是不可能完成本书的写作的。我们也十分感谢邵启满博士，他的许多宝贵意见使本书增色不少，我们也感谢陈斌、蔡宗武和苏中根等同志的有益的建议。

由于水平所限，书中有些结果不尽完善，不妥和谬误之处也一定不少，恳请同行专家和广大读者不吝赐教。

作　者

1990 年 7 月于杭州大学

# 目　录

# 第一章 Wiener 过程及其相关的 Gauss 过程的增量

Wiener过程和某些相关的Gauss过程增量的结果深入地刻划了样本轨道的性质．它们是近10余年来概率论中的重要成就．随着Csörgő和Révész的专著"Strong Approximations in Probability and Statistics"于1981年问世，许多作者对这一课题作了研究．在本章中，我们将介绍这一方面的若干新进展．

## § 1.1 Wiener过程的增量有多大?

### 1.1.1 Csörgő-Révész增量

设$\{W(t);0\leqslant t<\infty\}$是概率空间$(\Omega,\mathscr{F},P)$上的标准Wiener过程．Csörgő 和 Révész 首先证明下述定理．

**定理1.1.1** (Csörgő and Révész, 1979) 设 $0<a_T\leqslant T$是$T$的函数，满足：

(i) $a_T$是单调不减的；

(ii) $T/a_T$ 是单调不减的．

那么

$$(1.1.1)\qquad \varlimsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\beta_T|W(t+s)-W(t)|=1 \quad \text{a.s.}$$

$$(1.1.2)\qquad \varlimsup_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\beta_T|W(t+a_T)-W(t)|=1 \quad \text{a.s.}$$

$$(1.1.3)\qquad \varlimsup_{T\to\infty}\sup_{0\leqslant s\leqslant a_T}\beta_T|W(T+s)-W(T)|=1 \quad \text{a.s.}$$

$$(1.1.4)\qquad \varlimsup_{T\to\infty}\beta_T|W(T+a_T)-W(T)|=1 \quad \text{a.s.}$$

其中

$$\beta_T=\{2a_T(\log T/a_T+\log\log T)\}^{-1/2}\text{[1]}$$

若$a_T$还满足

$$\text{(iii)}\qquad \lim_{T\to\infty}(\log T/a_T)/\log\log T=\infty,$$

那么

$$(1.1.5)\quad \lim_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\beta_T|W(t+s)-W(t)|=1\quad \text{a.s.}$$

$$(1.1.6)\quad \lim_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\beta_T|W(t+a_T)-W(t)|=1\quad \text{a.s.}$$

Deo (1977) 指出：若条件 (iii) 不真，那么当$\overline{\lim\limits_{T\to\infty}}(\log T/a_T)/\log\log T<\infty$时，有

$$\underline{\lim_{T\to\infty}}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\beta_T|W(t+s)-W(t)|<1\quad \text{a.s.}$$

这就提出了一个问题：寻求正则化因子$\delta_T=\delta_T(a_T)$ 使得

$$(1.1.7)\quad \underline{\lim_{T\to\infty}}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\delta_T|W(t+s)-W(t)|=1\quad \text{a.s.}$$

$$(1.1.8)\quad \underline{\lim_{T\to\infty}}\sup_{0\leqslant t\leqslant T-a_T}\delta_T|W(t+a_T)-W(t)|=1\quad \text{a.s.}$$

关于(1.1.7)和(1.1.8) 的部分回答由Book和Shore(1978)，Csáki和Révész(1979)，邵启满(1986) 等给出．其中一些结果如下：

1．(BookandShore，1978) 设$a_T$如定理1.1.1.若

$$\text{(iv)}\qquad \lim_{T\to\infty}(\log T/a_T)/\log\log T=r\qquad 0\leqslant r\leqslant\infty,$$

那么

$$(1.1.9)\quad \lim_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\beta_T|W(t+a_T)-W(t)|=\left(\frac{r}{1+r}\right)^{1/2}\quad \text{a.s.}$$

2．(Csáki and Révész，1979) 设$a_T$如定理1.1.1.那么

$$18^{-1}\leqslant\underline{\lim_{T\to\infty}}\gamma_1(T)\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}|W(t+s)-W(t)|\leqslant 46\quad \text{a.s.}$$

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1) 此处及以后我们定义 $\log t=\log(\max(t,1))$，$\log\log t=\log\log(\max(t,e))$.

其中

$$\gamma_1(T)=\left\{2a_T\log\left(1+\frac{\pi^2}{16}\frac{T}{a_T\log\log T}\right)\right\}^{-1/2}.$$

进一步，若

(v) $\lim\limits_{T\to\infty}(\log T/a_T)/\log\log\log T=\infty,$

那么

(1.1.10) $\varlimsup\limits_{T\to\infty}\gamma_1(T)\sup\limits_{0\le t\le T-a_T}\sup\limits_{0\le s\le a_T}|W(t+s)-W(t)|=1$ a.s.

由邵启满（1986）减弱条件（V）得到下述结果

**定理1.1.2**（邵启满，1986） 设$a_T$如定理1.1.1.若

(vi) $\lim\limits_{T\to\infty}(T/a_T)/\log\log T=\infty,$

那么

(1.1.11) $\varliminf\limits_{T\to\infty}\gamma(T)\sup\limits_{0\le t\le T-a_T}|W(t+a_T)-W(t)|=1$ a.s.

(1.1.12) $\varliminf\limits_{T\to\infty}\gamma(T)\sup\limits_{0\le t\le T-a_T}\sup\limits_{0\le s\le a_T}|W(t+s)-W(t)|=1$

a.s.

其中

$$\gamma(T)=\{2a_T(\log T/a_T-\log\log\log T)\}^{-1/2}.$$

为证明这一定理，我们需要下述引理.

**引理1.1.1**（Slepian，1962；Adler，1989）设$\{X(t);t\in T\}$和$\{Y(t);t\in T\}$是Gauss过程，均值为0，对一切$t\in T$，$EX^2(t)=EY^2(t)$且对一切$s,t\in T$，$EY(t)Y(s)\le EX(t)X(s)$。那么

$$P\{\sup_{t\in T}X(t)\le u\}\ge P\{\sup_{t\in T}Y(t)\le u\}.$$

**引理1.1.2**（Révész，1982） 设$k$是任给正整数.那么对任一$\varepsilon>0$存在$u_0=u_0(\varepsilon)$使得对一切$u\ge u_0$有

$$\begin{aligned}(1.1.13)\quad &(1-\varepsilon)\frac{ku}{\sqrt{2\pi}}e^{-u^2/2}\\ &\le P\{\sup_{0\le x\le k}(W(x+1)-W(x))>u\}\\ &\le P\{\sup_{0\le x\le k}\sup_{0\le s\le 1}(W(x+s)-W(x))>u\}\le c\frac{ku}{\sqrt{2\pi}}e^{-u^2/2},\end{aligned}$$

其中常数 $c<25$.

**证** (1.1.13) 中第一个不等式是熟知的（见Qualls and Watanabe, 1972）.我们仅需证明后一不等式. 设

$$x_i = i/u^2,\ \ i=1,2,\cdots,[u^2k]^{1)}$$

是区间$[0,k]$的一个分划.定义事件

$$B_i=\{\sup_{0\leqslant s\leqslant u^{-2}}(W(x_i-s)-W(x_i))>1\},$$

$$\begin{aligned}A_i(v)=\{&\sup_{0\leqslant s\leqslant 1}(W(x_i+s)-W(x_i))\\ &\geqslant u-v/u,\quad (v-\Delta v)/u\\ &\leqslant \sup_{0\leqslant s\leqslant u^{-2}}(W(x_i-s)-W(x_i))\leqslant v/u\}.\end{aligned}$$

那么当$u\to\infty$且$\Delta v\to 0$时，我们有

$$\begin{aligned}P(A_i(v))\approx&\frac{2}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)\cdot\frac{2}{\sqrt{2\pi}}\frac{1}{u-v/u}\\ &\cdot\exp\left(-\frac{1}{2}\left(u-\frac{v}{u}\right)^2\right)\Delta v,\end{aligned}$$

$$P(B_i)\approx\frac{2}{u\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right).$$

所以存在一个绝对常数$u_0>0$使对一切$u>u_0$有

$$\begin{aligned}&P\{\sup_{0\leqslant x\leqslant k}\ \sup_{0\leqslant s\leqslant 1}(W(x+s)-W(x))>u\}\\ &\leqslant[u^2k]\left\{\frac{2}{\pi}\int_0^u\frac{1}{u-v/u}\exp\left(-\frac{v^2}{2}\right)\right.\\ &\qquad\left.\exp\left(-\frac{1}{2}\left(u-\frac{v}{u}\right)^2\right)\cdot dv+\frac{2}{u\sqrt{2\pi}}\exp\left(-\frac{u}{2}\right)^2\right\}\\ &\leqslant c\ \frac{ku}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right),\end{aligned}$$

其中$c<25$.引理1.1.2证毕.

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1）在本书中，记号[·]有时表示最大整数部分，有时表示括号，这不难从行文中看出.

**引理1.1.3** (Révész, 1982) 对任一$\varepsilon>0$存在$u_0=u_0(\varepsilon)>0$和$T_0=T_0(\varepsilon)>0$使得对一切$u\geqslant u_0$和$T\geqslant T_0$有

$$(1.1.14)\quad \exp\left\{-25\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2}\right\}$$

$$\leqslant P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant 1}(W(t+s)-W(t))\leqslant u\}$$

$$\leqslant P\{\sup_{0\leqslant t\leqslant T}(W(t+1)-W(t))\leqslant u\}$$

$$\leqslant \exp\left\{-(1-\varepsilon)\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2}\right\}.$$

**证** 设$k=[T]$.从引理1.1.1和引理1.1.2我们有

$$P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant 1}(W(t+s)-W(t))\leqslant u\}$$

$$\geqslant P\{\max_{0\leqslant i\leqslant k}\sup_{i\leqslant t\leqslant i+1}\sup_{0\leqslant s\leqslant 1}(W(t+s)-W(t))\leqslant u\}$$

$$\geqslant (1-P\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s\leqslant 1}(W(t+s)-W(t))>u\})^{k+1}$$

$$\geqslant\left(1-\frac{cu}{\sqrt{2\pi}}e^{-u^2/2}\right)^{k+1}$$

$$\geqslant\exp\left(-25\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2}\right)\qquad (当u>u_0),$$

这就证明了 (1.1.14) 中第一个不等式.现在我们来证(1.1.14)的后一不等式.若$k<T$是正整数，那么

$$\sup_{0\leqslant t\leqslant T}(W(t+1)-W(t))$$

$$\geqslant\max_{0\leqslant i\leqslant l}\sup_{i(k+1)\leqslant t<(i+1)(k+1)}(W(t+1)-W(t)),$$

其中$l$是使$(l+1)(k+1)-1\leqslant T$的最大整数.容易看出事件$\{\sup_{i(k+1)\leqslant t<(i+1)(k+1)}(W(t+1)-W(t)),\ i=0,1,\cdots,l\}$是相互独立的.因此由引理1.1.2我们有

$$P\{\sup_{0\leqslant t\leqslant T}(W(t+1)-W(t))\leqslant u\}$$

$$\leqslant P\{\max_{0\leqslant i\leqslant l}\sup_{i(k+1)\leqslant t<(i+1)(k+1)}(W(t+1)-W(t))\leqslant u\}$$

$$\leqslant(P\{\sup_{0\leqslant t\leqslant k}(W(t+1)-W(t))\leqslant u\})^{l+1}$$

$$\leqslant\left(1-\frac{1+o(1)}{\sqrt{2\pi}}kue^{-u^2/2}\right)^{l+1}$$

$$\leqslant\exp\left\{-(1+o(1))\frac{k(l+1)u}{\sqrt{2\pi}}e^{-u^2/2}\right\}$$

$$\leqslant\exp\left\{-(1-\varepsilon)\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2}\right\}\qquad (T\geqslant T_0(\varepsilon),u\geqslant u_0(\varepsilon)).$$

这就证明了引理1.1.3.

从引理1.1.3的证明即可写出

**引理1.1.4** (Révész, 1982) 对任给的$\varepsilon>0$存在$u_0=u_0(\varepsilon)$和$T_0=T_0(\varepsilon)$使对一切$T\geqslant T_0$和$u\geqslant u_0$有

$$\exp\left\{-50\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2}\right\}$$

$$\leqslant P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant 1}|W(t+s)-W(t)|\leqslant u\}$$

$$\leqslant P\{\sup_{0\leqslant t\leqslant T}|W(t+1)-W(t)|\leqslant u\}$$

$$\leqslant\exp\left\{-2(1-\varepsilon)\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2}\right\}.$$

我们还需要下述显明的结果

**引理1.1.5** 设$\{\xi,\xi_n;\ n\geqslant 1\}$是随机变量序列.若

$$P\{\xi_n\geqslant\xi\}\to 0\qquad (n\to\infty),$$

那么存在子列$\{\xi_{n_k}\}$使得

$$\overline{\lim_{k\to\infty}}\,\xi_{n_k}\leqslant\xi\qquad \text{a.s.}$$

所以

$$\underline{\lim_{n\to\infty}}\,\xi_n\leqslant\xi\qquad \text{a.s.}$$

**注1.1.1** 易见，若$P\{\xi_n\leqslant\xi\}\to 0(n\to\infty)$，就有

$$\overline{\lim_{n\to\infty}}\,\xi_n\geqslant\xi\qquad \text{a.s.}$$

定理1.1.2的证明

1° 我们来证

(1.1.15) $$\underline{\lim_{T\to\infty}}\,\gamma(T)\sup_{0\leqslant t\leqslant T-a_T}|W(t+a_T)-W(t)|\geqslant 1\quad \text{a.s.}$$

在引理1.1.4中取$\varepsilon=1/2$，对充分大的$T$我们有

$$P\{\sup_{0\le t\le T-a_T}|W(t+a_T)-W(t)|<\gamma^{-1}(T)\}$$

$$\le \exp\left\{-\frac{T}{a_T\sqrt{2\pi}}\sqrt{2\log\frac{T}{a_T\log\log T}}\frac{a_T\log\log T}{T}\right\}$$

$$\le(\log T)^{-1}.$$

记 $T_k=k^{\nu k}(k=1,2,\cdots)$.由Borel- Cantelli 引理我们得

$$(1.1.16)\quad \varliminf_{k\to\infty}\gamma(T_k)\sup_{0\le t\le T_k-a_{T_k}}|W(t+a_{T_k})-W(t)|\ge 1\ \text{a.s.}$$

当$T_k\le T\le T_{k+1}$时，我们有

$$(1.1.17)\quad \gamma(T)\sup_{0\le t\le T-a_T}|W(t+a_T)-W(t)|$$

$$\ge\left(2a_{T_{k+1}}\log\frac{T_{k+1}}{a_{T_{k+1}}\log\log T_k}\right)^{-1/2}$$

$$\left(\sup_{0\le t\le T_k-a_{T_k}}|W(t+a_{T_k})-W(t)|\right.$$

$$\left.-\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_{k+1}}-a_{T_k}}|W(t+s)-W(t)|\right)$$

$$=:A_k\gamma(T_k)I(T_k)-Z_k(T_k),$$

其中

$$A_k=\left(\frac{T_k}{T_{k+1}}\cdot\frac{(a_{T_k}/T_k)\log((T_k/a_{T_k})/\log\log T_k)}{(a_{T_{k+1}}/T_{k+1})\log((T_{k+1}/a_{T_{k+1}})/\log\log T_k)}\right)^{1/2}.$$

注意到当eax$\le1,a>0$时$x\log(1/xa)$是$x$的单调增函数，所以

$$(1.1.18)\qquad 1\ge\lim_{k\to\infty}A_k\ge\lim_{k\to\infty}(T_k/T_{k+1})^{1/2}=1.$$

另一方面，由定理1.1.1有

$$\varlimsup_{k\to\infty}\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_{k+1}}-a_{T_k}}\beta_1(T_k)|W(t+s)-W(t)|$$

$$\le 1\quad \text{a.s.}$$

其中

$$\beta_1(T_k)=\left\{2\left(a_{T_{k+1}}-a_{T_k}\right)\left(\log\frac{T_k+a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}\right.\right.$$

$$+\log\log(T_k+a_{T_{k+1}})\Big)\Big\}^{-1/2}.$$

易见对充分大的$k$我们有

$$a_{T_{k+1}}-a_{T_k}\leqslant a_{T_{k+1}}(1-T_k/T_{k+1})\leqslant 6a_{T_{k+1}}/k^{1/3},$$

由此可得

$$\beta_1^{-2}(T_k)(2a_{T_{k+1}}\log((T_{k+1}/a_{T_{k+1}})/\log\log T_k))^{-1}$$

$$\leqslant\frac{6}{k^{1/3}}\Big(\log\Big(\frac{T_{k+1}+a_{T_{k+1}}}{6a_{T_{k+1}}}k^{1/3}\Big)$$

$$+\log\log(T_k+a_{T_{k+1}})\Big)\Big/\log\Big(\frac{T_{k+1}/a_{T_{k+1}}}{\log\log T_k}\Big)$$

$$\to 0\qquad (k\to\infty).$$

所以有

$$\overline{\lim_{k\to\infty}}\ Z_k(T_k)=0\qquad \text{a.s.}$$

由它及(1.16)—(1.18)，得证(1.15) 成立。

2° 我们来证

(1.1.19) $$\overline{\lim_{T\to\infty}}\ \gamma(T)\sup_{0\leqslant t\leqslant T-a_T}\ \sup_{0\leqslant s\leqslant a_T}|W(t+s)-W(t)|\leqslant 1\ \text{a.s.}$$

若$\overline{\lim\limits_{T\to\infty}}\ (\log T/a_T)/\log\log\log T=\infty$，此时存在正数列$\{T_N\}$ 使得

(1.1.20) $$\lim_{N\to\infty}\ (\log T_N/a_{T_N})/\log\log\log T_N=\infty$$

对任给 $\varepsilon>0$，利 用Csörgő和Révész (1981) 中引理1.2.1，我们有

$$P\{(2a_{T_N}\log T_N/a_{T_N})^{-1/2}\sup_{0\leqslant t\leqslant T_N-a_{T_N}}\ \sup_{0\leqslant s\leqslant a_{T_N}}|W(t+s)$$

$$-W(t)|\geqslant 1+\varepsilon\}$$

$$\leqslant C\frac{T_N}{a_{T_N}}\exp\Big\{-(1+\varepsilon)\log\frac{T_N}{a_{T_N}}\Big\}=C\Big(\frac{a_{T_N}}{T_N}\Big)^{\varepsilon}\to 0(N\to\infty).$$

因此从引理1.1.5我们有

$$\lim_{N\to\infty}\ (2a_{T_N}\log T_N/a_{T_N})^{-1/2}\sup_{0\leqslant t\leqslant T_N-a_{T_N}}\ \sup_{0\leqslant s\leqslant a_{T_N}}|W(t+s)$$

$$-W(t)|\leqslant 1 \qquad \text{a.s.}$$

而从 (1.1.20) 知

$$\lim_{N\to\infty}(2a_{T_N}\log T_N/a_{T_N})^{-1/2}/\gamma(T_N)=1.$$

因此这时(1.1.19)成立.

若$\overline{\lim\limits_{T\to\infty}}(\log T/a_T)/\log\log\log T<\infty$，即存在常数$C_0>0$，使得

$$T/a_T\leqslant(\log\log T)^{C_0}. \tag{1.1.21}$$

设$T_k=e^{k^2}(k=2,\ 3,\ \cdots)$. 利用引理1.1.4及条件(vi)，对任给$\varepsilon>0$，当$k$充分大时我们有

$$\begin{aligned}&P\{\sup_{T_k\leqslant t\leqslant T_{k+1}-a_{T_{k+1}}}\sup_{0\leqslant s\leqslant a_{T_{k+1}}}|W(t+s)-W(t)|\\&\leqslant(1+\varepsilon)\gamma^{-1}(T_{k+1})\}\\&\geqslant\exp\Big\{-\frac{100}{\sqrt{2\pi}}\cdot\frac{T_{k+1}-T_k}{a_{T_{k+1}}}\\&\quad\cdot\frac{(1+\varepsilon)\sqrt{2\log(T_{k+1}/a_{T_{k+1}}\log\log T_{k+1})}}{(T_{k+1}/a_{T_{k+1}})^{1+\varepsilon}}(\log\log T_{k+1})^{1+\varepsilon}\Big\}\\&\geqslant k^{-2/3}.\end{aligned}$$

由Borel-Cantelli 引理即得

$$\begin{aligned}\lim_{k\to\infty}\gamma(T_{k+1})\sup_{T_k\leqslant t\leqslant T_{k+1}-a_{T_{k+1}}}\sup_{0\leqslant s\leqslant a_{T_{k+1}}}&|W(t+s)\\-W(t)|\leqslant 1+\varepsilon &\qquad \text{a.s.}\end{aligned} \tag{1.1.22}$$

注意到

$$\begin{aligned}\sup_{0\leqslant t\leqslant T_{k+1}-a_{T_{k+1}}}\sup_{0\leqslant s\leqslant a_{T_{k+1}}}&|W(t+s)-W(t)|\\\leqslant\sup_{T_k\leqslant t\leqslant T_{k+1}-a_{T_{k+1}}}\sup_{0\leqslant s\leqslant a_{T_{k+1}}}&|W(t+s)-W(t)|\\+\sup_{0\leqslant u<v\leqslant T_k}&|W(v)-W(u)|.\end{aligned} \tag{1.1.23}$$

由熟知的重对数律

$$\overline{\lim_{k\to\infty}} \sup_{0\leqslant u<v\leqslant T_k} (2T_k\log\log T_k)^{-1/2}|W(v)-W(u)|\leqslant 1 \quad \text{a.s.}$$

且从 (1.1.21) 和条件 (vi)，我们有

$$\gamma(T_{k+1})(2T_k\log\log T_k)^{-1/2}\to 0 \qquad (k\to\infty).$$

所以

$$(1.1.24)\quad \overline{\lim_{k\to\infty}}\ \gamma(T_{k+1}) \sup_{0\leqslant u<v\leqslant T_k}|W(v)-W(u)|=0 \quad \text{a.s.}$$

故综合 (1.22)—(1.24)得证 (1.19)成立.定理证毕.

### 1.1.2 滞后增量

Wiener 过程的另一形式增量——滞后 (lag)增量是由Hanson 和 Russo (1983) 提出并讨论的. 陈桂景、孔繁超和林正炎(1986)强化了他们的结果并证明

**定理1.1.3** (Chen, Kong and Lin, 1986)

$$(1.1.25)\quad \overline{\lim_{T\to\infty}} \sup_{0<t\leqslant T}|W(T)-W(T-t)|/d(T,t)=1 \quad \text{a.s.}$$

$$(1.1.26)\quad \lim_{T\to\infty} \sup_{0<t\leqslant T} \sup_{t\leqslant s\leqslant T}|W(s)-W(s-t)|/d(T,t)=1 \quad \text{a.s.}$$

$$(1.1.27)\quad \overline{\lim_{T\to\infty}} \sup_{0<t\leqslant T} \sup_{0\leqslant s\leqslant t}|W(T)-W(T-s)|/d(T,t)=1 \quad \text{a.s.}$$

其中

$$d(T,t)=\{2t(\log Tt^{-1}+\log\log t)\}^{1/2}.$$

**证** 1° 由重对数律我们有

$$(1.1.28)\quad (1.25)\text{ 式的左边}\geqslant \overline{\lim_{T\to\infty}}|W(T)|/(2T\log\log T)^{1/2} = 1 \quad \text{a.s.}$$

2°为证明

$$(1.1.29)\quad \overline{\lim_{T\to\infty}} \sup_{0<t\leqslant T} \sup_{t\leqslant s\leqslant T}|W(s)-W(s-t)|/d(T,t)\leqslant 1 \quad \text{a.s.}$$

我们取实数$\theta>1$使得对某$\varepsilon>0$有$1<2(1+\varepsilon)^2/((2+\varepsilon)\theta)=:1+2\varepsilon'$。对$n=1, 2, \cdots$和$k=\cdots, -1, 0, 1, 2, \cdots, k_n$记$T_n=2^n$，$t_k=\theta^k$，其中$k_n=[(n+1)\log 2/\log\theta]+1$。写$k_0=[1/\log\theta]$，$k_n'=[\log(T_{n+1}/(\log T_n)^{1/\varepsilon'})/\log\theta]$。

当$T_n\leqslant T\leqslant T_{n+1}$时，我们有

$$(1.1.30)\quad \sup_{0\leqslant t\leqslant T}\sup_{t\leqslant s\leqslant T}|W(s)-W(s-t)|/d(T, t)$$

$$\leqslant \sup_{-\infty<k\leqslant k_n-1}\sup_{t_k\leqslant t\leqslant t_{k+1}}\sup_{t\leqslant s\leqslant T_{n+1}}\frac{|W(s)-W(s-t)|}{\{2t_k(\log(T_n/t_{k+1})+\log\log t_k)\}^{1/2}}$$

$$=:\sup_{-\infty<k\leqslant k_n-1}A_{nk}.$$

检查Csörgő和Révész（1981）中引理1.1.1和1.2.1的证明可见对任一$T>0$，$v>0$和$0<h\leqslant T$，我们有

$$(1.1.31)\quad P\left\{\sup_{0\leqslant s'\leqslant s\leqslant T,\ s-s'\leqslant h}h^{-1/2}|W(s)-W(s')|\geqslant v\right\}$$

$$\leqslant\frac{cT}{h}\exp\left\{-\frac{v^2}{2+\varepsilon}\right\},$$

其中$c$是仅依赖于$\varepsilon$的正常数。对于$-\infty<k\leqslant k_0$利用这一不等式，我们有

$$(1.1.32)\ P(A_{nk}\geqslant 1+\varepsilon)$$

$$\leqslant P\left\{\sup_{t\leqslant s-t,\ s\leqslant T_{n+1}}\sup_{0<t\leqslant t_{k+1}}t_{k+1}^{-\frac{1}{2}}|W(s)-W(s-t)|\right.$$

$$\left.\geqslant(1+\varepsilon)\left(\frac{2t_k}{t_{k+1}}\log\frac{T_n}{t_{k+1}}\right)^{\frac{1}{2}}\right\}$$

$$\leqslant c\left(\frac{T_{n+1}}{t_{k+1}}\right)\exp\left\{-2\left(\frac{(1+\varepsilon)^2}{(2+\varepsilon)\theta}\right)\log\left(\frac{T_n}{t_{k+1}}\right)\right\}^{1)}$$

1）这里及以后$c$是指正的常数，它在不同的地方可取不同的值。

$$= c\frac{T_{n+1}}{t_{k+1}}\left(\frac{t_{k+1}}{T_n}\right)^{1+2\varepsilon'} = c(\theta^{k+1}2^{-n})^{2\varepsilon'}.$$

由此即得

$$(1.1.33)\qquad \sum_{n=1}^{\infty}\sum_{-\infty<k\leqslant k_\theta} P(A_{nk}\geqslant 1+\varepsilon)$$

$$\leqslant c\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(\theta^{-k}2^{-n})^{2\varepsilon'} + c\sum_{n=1}^{\infty}(k_\theta+1)(\theta e)^{2\varepsilon'}2^{-2n\varepsilon'}$$

$$<\infty.$$

对于$k_\theta<k\leqslant k_n-1$情形，再利用不等式（1.1.31），我们有

$$(1.1.34)\qquad P(A_{nk}\geqslant 1+\varepsilon)$$

$$\leqslant P\left\{\sup_{0\leqslant s-t,s\leqslant T_{n+1}}\ \sup_{0\leqslant t\leqslant t_{k+1}} t_{k+1}^{-1/2}|W(s)-W(s-t)|\right.$$

$$\left.\geqslant(1+\varepsilon)\left(\frac{2t_k}{t_{k+1}}\left(\log\frac{T_n}{t_{k+1}}+\log\log t_k\right)\right)^{\frac{1}{2}}\right\}$$

$$\leqslant c\frac{T_{n+1}}{t_{k+1}}\exp\left\{-\frac{2(1+\varepsilon)^2}{(2+\varepsilon)\theta}\log\frac{T_n\log t_k}{t_{k+1}}\right\}$$

$$= c(t_{k+1}/T_{n+1})^{2\varepsilon'}(\log t_k)^{-(1+2\varepsilon')}.$$

注意到当$k_\theta<k\leqslant k_n'$时，我们有

$$(t_{k+1})^{2\varepsilon'}\leqslant\theta^{(k_n'+1)2\varepsilon'}\leqslant\left(\frac{\theta T_{n+1}}{(\log T_n)^{1/\varepsilon'}}\right)^{2\varepsilon'} = \frac{\theta^{2\varepsilon'}T_{n+1}^{2\varepsilon'}}{(\log T_n)^2}.$$

故由（1.1.34）即得

$$(1.1.35)\qquad \sum_{n=1}^{\infty}\sum_{k=k_\theta+1}^{k_n'} P(A_{nk}\geqslant 1+\varepsilon)$$

$$\leqslant c\sum_{n=1}^{\infty}\frac{\theta^{2\varepsilon'}}{(\log T_n)^2}\sum_{k=k_\theta+1}^{k_n'}(\log t_k)^{-1-2\varepsilon'}$$

$$\leqslant c\sum_{n=1}^{\infty}n^{-2}\sum_{k=1}^{\infty}k^{-1-2\varepsilon'}<\infty.$$

对$k_n' < k \leqslant k_n - 1$情形，我们有

$$T_n^{1/2} \leqslant t_{k+1} \leqslant \theta T_{n+1},$$

$$k_n - k_n' \leqslant (\varepsilon' \log\theta)^{-1} \log\log 2^n + 2 =: k_n''.$$

再应用不等式（1.1.34）我们得

$$(1.1.36) \quad \sum_{n=1}^{\infty} \sum_{k=k_n'+1}^{k_n-1} P(A_{nk} \geqslant 1+\varepsilon)$$

$$= c\sum_{n=1}^{\infty} \sum_{k=k_n'+1}^{k_n-1} \left(\frac{\theta T_{n+1}}{T_n}\right)^{2\varepsilon'} (\log T_n)^{-1-2\varepsilon'}$$

$$\leqslant c\sum_{n=1}^{\infty} k_n''(2\theta)^{2\varepsilon'} n^{-1-2\varepsilon'} \leqslant c\sum_{n=1}^{\infty} n^{-1-\varepsilon'} < \infty.$$

最后，综合（1.1.33），（1.1.35）和（1.1.36）得

$$\sum_{n=1}^{\infty} P\left(\sup_{-\infty<k\leqslant k_n-1} A_{nk} \geqslant 1+\varepsilon\right)$$

$$\leqslant \sum_{n=1}^{\infty} \sum_{-\infty<k\leqslant k_n-1} P(A_{nk} \geqslant 1+\varepsilon) < \infty.$$

由Borel-Cantelli引理即得（1.1.29）式成立.

从（1.1.28）和（1.1.29）得证（1.1.25）式成立.

3°为证（1.1.26）式，只需证明

$$(1.1.37) \quad \lim_{T\to\infty} \sup_{0<t\leqslant T} \sup_{t\leqslant s\leqslant T} |W(s) - W(s-t)| / d(T,t) \geqslant 1 \quad \text{a.s.}$$

记

$$B_n = \sup_{1\leqslant s\leqslant n} |W(s) - W(s-1)| / (2\log n)^{1/2}.$$

运用熟知的概率不等式：对$x>0$有

$$(1.1.38) \quad \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right) \leqslant P(W(1) \geqslant x)$$

$$\leqslant \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right),$$

即得

$$\sum_{n=2}^{\infty} P(B_n \leqslant 1-\varepsilon)$$

$$\leqslant \sum_{n=2}^{\infty} P\left\{ \max_{1\leqslant i\leqslant n} |W(i)-W(i-1)| \leqslant (1-\varepsilon)(2\log n)^{\frac{1}{2}} \right\}$$

$$= \sum_{n=2}^{\infty} \left\{ 1 - \frac{c}{(\log n)^{1/2}} \left(\frac{1}{n}\right)^{(1-\varepsilon)^2} \right\}^n$$

$$\leqslant \sum_{n=2}^{\infty} \exp\left\{ -\frac{cn}{(\log n)^{1/2}} \left(\frac{1}{n}\right)^{(1-\varepsilon)^2} \right\} < \infty,$$

故由Borel-Cantelli 引理，我们有$\varliminf_{n\to\infty} B_n \geqslant 1$ a.s. 注意到当$n \leqslant T \leqslant n+1$时

$$\sup_{0<t\leqslant T} \sup_{t\leqslant s\leqslant T} |W(s)-W(s-t)|/d(T, t)$$

$$\geqslant \sup_{1\leqslant s\leqslant T} |W(s)-W(s-1)|/(2\log T)^{1/2}$$

$$\geqslant B_n(\log n/\log(n+1))^{1/2},$$

所以，(1.1.37) 式得证.

注意到

(1.1.25) 式左边⩽ (1.1.27) 式左边⩽(1.1.26)式左边，从 (1.1.25) 和 (1.1.26) 得 (1.1.27) 式成立.定理证毕.

陈桂景、孔繁超和林正炎 (1986) 指出定理1.1.3 可被改写为如下一般形式

**定理1.1.3′** 设$0<a_T\leqslant T$，有

(1.1.25′) $$\varlimsup_{T\to\infty} \sup_{a_T\leqslant t\leqslant T} |W(T)-W(T-t)|/d(T, t)=1 \quad \text{a.s.}$$

(1.1.26′) $$\varlimsup_{T\to\infty} \sup_{a_T\leqslant t\leqslant T} \sup_{t\leqslant s\leqslant T} |W(s)-W(s-t)|/d(T,t)=1 \quad \text{a.s.}$$

(1.1.26″) $$\varlimsup_{T\to\infty} \sup_{a_T\leqslant t\leqslant T} \sup_{t\leqslant s\leqslant T} \sup_{0\leqslant h\leqslant t} |W(s)-W(s-h)|/d(T, t)=1 \quad \text{a.s.}$$

其中 (1.1.26″) 没有被陈桂景等 (1986) 提到，它可以仿照

(1.1.26′) 证明之.

**推论1.1.1** (Hanson and Russo, 1983a)

$$(1.1.39)\quad \lim_{a\to\infty}\ \sup_{\substack{0\leqslant u\leqslant s\leqslant t\leqslant v\\ a\leqslant v-u}} |W(t)-W(s)|/d(v,\ v-u)=1 \quad \text{a.s.}$$

**证** 注意到

$$(1.1.40)\quad \overline{\lim_{a\to\infty}}\ \sup_{0\leqslant u\leqslant s\leqslant t\leqslant v,\, a\leqslant v-u} |W(t)-W(s)|/d(v,\ v-u)$$

$$=\lim_{a_0\to\infty}\ \sup_{0\leqslant u\leqslant s\leqslant t\leqslant v,\ a_0\leqslant a\leqslant v-u}' |W(t)-W(s)|/d(v,v-u)$$

$$=\lim_{a\to\infty}\ \sup_{0\leqslant u\leqslant s\leqslant t\leqslant v,\, a\leqslant v-u} |W(t)-W(s)|/d(v,\ v-u),$$

所以(1.1.39)式中极限存在(可能为$\infty$).

假设$\omega$使得(1.1.27)式中"$\overline{\lim}$"等于1.固定这一$\omega$,并设$\varepsilon>0$.选取$T_0$使$T_0\geqslant e^e$且使

$$(1.1.41)\quad \sup_{T_0\leqslant T}\ \sup_{0<t\leqslant T}\ \sup_{0\leqslant s\leqslant t} |W(T)-W(T-s)|/d(T,t)\leqslant 1+\varepsilon.$$

令 $\tau=\min(v-u,\ t)$,我们有

$$(1.1.42)\quad \sup_{0\leqslant u\leqslant s\leqslant t\leqslant v,\ a_0\leqslant a\leqslant v-u}\ \sup_{T_0\leqslant t} |W(t)-W(s)|/d(v,\ v-u)$$

$$\leqslant \sup_{T_0\leqslant t}\ \sup_{a_0\leqslant v-u}\ \sup_{0\leqslant t-s\leqslant v-u,\ s\leqslant t} |W(t)-W(s)|/d(v,v-u)$$

$$\leqslant \sup_{T_0\leqslant t}\ \sup_{0<\tau\leqslant t}\ \sup_{0\leqslant t-s\leqslant \tau} |W(t)-W(t-(t-s))|/d(t,\tau)$$

$$\leqslant 1+\varepsilon.$$

另外,$\sup\limits_{0\leqslant s\leqslant t\leqslant T_0}|W(t)-W(s)|$ 是有限的,且当$v-u\to\infty$时$d(v,\ v-u)\to\infty$关于$v$一致地成立,所以

$$(1.1.43)\quad \overline{\lim_{a\to\infty}}\ \sup_{0\leqslant u\leqslant s\leqslant t\leqslant v,\ a\leqslant v-u,\ t\leqslant T_0} |W(t)-W(s)|/d(v,v-u)$$

$$=0 \qquad \text{a.s.}$$

结合(1.1.42)和(1.1.43)并注意到$\varepsilon$的任意性,我们得(1.1.40)$\leqslant 1$ a.s. 令$u=s=0$, $t=v=a$ 并应用重对数律, 可得相反的不等式.得证推论成立.

**注1.1.2** 推论1.1.1蕴含着下述结果(Hanson and Russo, 1983a)：假设$0<a_T\leqslant T$ 且$\lim\limits_{T\to\infty} a_T=\infty$。那么

$$(1.1.44)\quad \overline{\lim_{T\to\infty}} \sup_{0\leqslant u<v\leqslant T,\, a_T\leqslant v-u} |W(v)-W(u)|/d(v,v-u)=1 \quad \text{a.s.}$$

$$(1.1.45)\quad \overline{\lim_{T\to\infty}} \sup_{0\leqslant u\leqslant s\leqslant t\leqslant v\leqslant T,\, a_T\leqslant v-u} |W(t)-W(s)|/d(v,v-u) =1 \quad \text{a.s.}$$

### 1.1.3 增量的一般形式

我们在此将给出Wiener过程增量的一个一般形式，Csörgő-Révész增量和一类滞后增量都是这一一般形式的特殊情形，且定理1.1.1中关于$a_T$的条件被去掉了。这个一般形式首先由郤启满(1989)所讨论。

**定理1.1.4** 设$a_T$，$b_T$和$c_T$是$T$的非负函数，且满足$a_T+b_T\geqslant c_T\to\infty$ $(T\to\infty)$.若存在常数$A>0$使对任一$T>1$有

$$(1.1.46)\quad b_T-b_{T-1}\leqslant Aa_T,\qquad a_T+b_T\leqslant A(a_{T-1}+b_{T-1}).$$

那么

$$(1.1.47)\quad \overline{\lim_{T\to\infty}} \sup_{0\leqslant t} \sup_{0<s} \sup_{0\leqslant r\leqslant s} |W(t+r)-W(t)|/d(t+s\vee c_T,s) =1 \quad \text{a.s.}$$

$$(1.1.48)\quad \overline{\lim_{T\to\infty}} \beta(a_T+b_T,\ a_T)|W(a_T+b_T)-W(b_T)|=1 \quad \text{a.s.}$$

其中

$$\beta(M,\ m)=\{2m(\log M/m+\log\log M)\}^{-1/2}.$$

进一步，若对任一$0<\varepsilon<1$

$$(1.1.49)\quad \sum_{N=1}^{\infty} \exp\{-b_N/a_N^{\varepsilon}((a_N+b_N)\log(a_N+b_N))^{1-\varepsilon}\}<\infty,$$

且

$$(1.1.50)\qquad \lim_{T\to\infty} b_T/b_{[T]}=\lim_{T\to\infty} a_T/a_{[T]}=1,$$

那么

$$(1.1.51)\lim_{T\to\infty}\sup_{0\leqslant t}\sup_{0<s}\sup_{0\leqslant r\leqslant s}|W(t+r)-W(t)|/d(t+s\vee c_T,s)=1\quad\text{a.s.}$$

$$(1.1.52)\quad\lim_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\beta(t+a_T,a_T)|W(t+a_T)-W(t)|=1\quad\text{a.s.}$$

**证** 首先，我们证明

$$(1.1.53)\overline{\lim_{T\to\infty}}\sup_{0\leqslant t}\sup_{0<s}\sup_{0\leqslant r\leqslant s}|W(t+r)-W(t)|/d(t+s\vee c_T,s)\leqslant 1\quad\text{a.s.}$$

显然地，我们可以假设当$T\to\infty$时，$c_T$不减地趋于$\infty$，不然的话，我们可以代替$c_T$来考察$c_T^*=\inf\limits_{T\leqslant t}c_t$。

对任给的$B>0$，我们有

$$\begin{aligned}&\sup_{0\leqslant t}\sup_{0<s}\sup_{0\leqslant r\leqslant s}|W(t+r)-W(t)|/d(t+s\vee c_T,s)\\&=(\sup_{0\leqslant t}\sup_{B<s}\sup_{0\leqslant r\leqslant s}|W(t+r)-W(t)|/d(t+s\vee c_T,s))\\&\vee(\sup_{0\leqslant t}\sup_{0<s\leqslant B}\sup_{0\leqslant r\leqslant s}|W(t+r)-W(t)|/d(t+s\vee c_T,s))\\&=:I_1+I_2.\end{aligned}$$

由推论 1.1.1，对任给$\varepsilon>0$存在$B=B(\varepsilon)$使得$I_1\leqslant 1+\varepsilon$ a.s.设$\theta>1$，利用(1.1.31)式，对充分大的$T$，我们有

$$\begin{aligned}(1.1.54)&P\{I_2\geqslant 1+\varepsilon\}\\&\leqslant\sum_{i=-\infty}^{\infty}\sum_{j=0}^{\infty}P\Big\{\sup_{\theta^i\leqslant t<\theta^{i+1}}\sup_{B\theta^{-(j+1)}<s\leqslant B\theta^{-j}}\sup_{0\leqslant r\leqslant s}|W(t+r)-W(t)|/\sqrt{s}\\&\qquad\geqslant(1+\varepsilon)\Big(2\Big(\log\frac{\theta^i+c_T}{B\theta^{-j}}+\log\log B\theta^{-j-1}\Big)\Big)^{1/2}\Big\}\\&\leqslant c\sum_{i=-\infty}^{\infty}\sum_{j=0}^{\infty}\theta^i(\theta^i+c_T)^{-(1+\varepsilon)}\theta^{-j\varepsilon/2}\leqslant cc_T^{-\varepsilon}\to 0\\&(T\to\infty).\end{aligned}$$

由于$I_2=I_2(T)$是$T$的不增函数，$\overline{\lim\limits_{T\to\infty}}\ I_2\leqslant 1+\varepsilon$ a.s.

这就证明了(1.1.53)成立。

其次，为证(1.1.47)和(1.1.48)式，我们仅需证明对任给的 $0<\varepsilon<1/8$ 有

(1.1.55) $\overline{\lim\limits_{N\to\infty}}\ \beta(a_N+b_N,a_N)|W(a_N+b_N)-W(b_N)|\geqslant 1-2\varepsilon$ a.s.

定义 $N_1=1$,

$N_{k+1}=\min\{n:\ n>N_k,\ b_n+\varepsilon^2a_n\geqslant b_{N_k}+a_{N_k}\}\quad k\geqslant 1$,

这就是说，对任一 $k\geqslant 1$ 和 $n<N_{k+1}$，我们有

(1.1.56) $b_{N_{k+1}}+\varepsilon^2a_{N_{k+1}}\geqslant b_{N_k}+a_{N_k},\quad b_n+\varepsilon^2a_n<b_{N_k}+a_{N_k}$.

容易看出：$N_{k+1}>N_k,\ b_{N_{k+1}}+a_{N_{k+1}}>b_{N_k}+a_{N_k},\ k\geqslant 1$.

我们来证明

(1.1.57) $$\sum_{k=1}^{\infty} a_{N_k}/(b_{N_k}+a_{N_k})\log\ (b_{N_k}+a_{N_k})=\infty.$$

从 (1.1.56) 和 (1.1.46)，我们有

(1.1.58) $$b_{N_{k-1}}+a_{N_{k-1}}\geqslant b_{N_k-1}\geqslant b_{N_k}-Aa_{N_k}$$
$$=b_{N_k}+a_{N_k}-(A+1)a_{N_k},$$

(1.1.59) $$b_{N_{k-1}}+a_{N_{k-1}}\geqslant\varepsilon^2(b_{N_k-1}+a_{N_k-1})\geqslant\varepsilon^2(b_{N_k}+a_{N_k})/A.$$

那么应用 (1.1.58) 和 (1.1.59)式，我们得

$$\sum_{k=2}^{\infty}\frac{a_{N_k}}{(b_{N_k}+a_{N_k})\log(b_{N_k}+a_{N_k})}$$
$$\geqslant\frac{1}{A+1}\sum_{k=2}^{\infty}\frac{(b_{N_k}+a_{N_k})-(b_{N_{k-1}}+a_{N_{k-1}})}{(b_{N_k}+a_{N_k})\log(b_{N_k}+a_{N_k})}$$
$$\geqslant\frac{\varepsilon^2}{A(A+1)}\sum_{k=2}^{\infty}\ \frac{b_{N_k}+a_{N_k}-(b_{N_{k-1}}+a_{N_{k-1}})}{(b_{N_{k-1}}+a_{N_{k-1}})\log(A(b_{N_{k-1}}+a_{N_{k-1}})/\varepsilon^2)}$$
$$\geqslant\frac{\varepsilon^2}{A(A+1)}\sum_{k=2}^{\infty}\ \int_{b_{N_{k-1}}+a_{N_{k-1}}}^{b_{N_k}+a_{N_k}}\ \frac{1}{x\log(Ax/\varepsilon^2)}dx=\infty.$$

这就得证 (1.1.57) 式成立.

记 $G=\{k:b_{N_k}\geqslant b_{N_{k-1}}+a_{N_{k-1}}\},H=\{k:b_{N_k}<b_{N_{k-1}}+a_{N_{k-1}}\}$.

情形1 假设

$$(1.1.60)\qquad \sum_{k\in G} a_{N_k}/(b_{N_k}+a_{N_k})\log(b_{N_k}+a_{N_k})=\infty.$$

记

$$\beta(k)=\beta(b_{N_k}+a_{N_k},\ a_{N_k}),$$

$$A_k=\{\beta(k)|W(b_{N_k}+a_{N_k})-W(b_{N_k})|\geqslant 1-\varepsilon\}.$$

注意到$A_k$，$k\in G$ 是独立的，所以为证（1.1.55）成立，我们只需证明

$$(1.1.61)\qquad \sum_{k\in G} P(A_k)=\infty.$$

对于充分大的$k\in G$，我们有

$$P(A_k)=P\{\beta(k)|W(a_{N_k})|\geqslant 1-\varepsilon\}$$

$$\geqslant c\exp\{-(1-\varepsilon)(\log((b_{N_k}+a_{N_k})/a_{N_k})+\log\log(b_{N_k}+a_{N_k}))\}$$

$$\geqslant ca_{N_k}/((b_{N_k}+a_{N_k})\log(b_{N_k}+a_{N_k})).$$

因此从假设（1.1.60）就得（1.1.61）式成立。

情形2 若（1.1.60）不真，那么由（1.1.57）得

$$(1.1.62)\qquad \sum_{k\in H} a_{N_k}/((b_{N_k}+a_{N_k})\log(b_{N_k}+a_{N_k}))=\infty.$$

对任一$k\in H$，从（1.1.56）我们有

$$(1.1.63)\qquad 0\leqslant b_{N_{k-1}}+a_{N_{k-1}}-b_{N_k}\leqslant\varepsilon^2 a_{N_k}$$

和

$$(1.1.64)\qquad (1-\varepsilon^2)a_{N_k}\leqslant b_{N_k}+a_{N_k}-(b_{N_{k-1}}+a_{N_{k-1}})\leqslant a_{N_k}.$$

注意到

$$(1.1.65)\qquad \beta(k)|W(b_{N_k}+a_{N_k})-W(b_{N_k})|$$

$$\geqslant\beta(k)\{|W(b_{N_k}+a_{N_k})-W(b_{N_{k-1}}+a_{N_{k-1}})|$$

$$-|W(b_{N_{k-1}}+a_{N_{k-1}})-W(b_{N_k})|\}.$$

类似于有关(1.1.53)的讨论，我们有

$$(1.1.66)\qquad \overline{\lim_{k\in H,k\to\infty}}\ \beta(k)|W(b_{N_{k-1}}+a_{N_{k-1}})-W(b_{N_k})|\leqslant\varepsilon\quad \text{a.s.}$$

现在为证(1.1.55)成立，我们仅需证明

(1.1.67) $$\overline{\lim_{k\in H, k\to\infty}}\ \beta(k)|W(b_{N_k}+a_{N_k})-W(b_{N_{k-1}}+a_{N_{k-1}})|$$

$$\geqslant 1-\varepsilon \qquad \text{a.s.}$$

仿照情形 1 的 证 明，就得(1.1.67)式。这就证 明 了(1.1.47)和(1.1.48)成立。

最后，假设条件(1.1.49)和(1.1.50)被满足，我们来证

(1.1.68) $$P\{\max_{0\leqslant j\leqslant [b_N/a_N]}\beta(b_N+a_N,a_N)|W((j+1)a_N)-W(ja_N)|$$

$$\leqslant 1-\varepsilon \text{ i.o.}\}=0.$$

由于$|W((j+1)a_N)-W(ja_N)|$，$j=1, 2, \cdots, [b_N/a_N]$是独立的，应用 (1.1.38) 式我们得

$$P\{\max_{0\leqslant j\leqslant [b_N/a_N]}\beta(b_N+a_N, a_N)|W((j+1)a_N)-W(ja_N)|$$

$$\leqslant 1-\varepsilon\}$$

$$\leqslant\left(1-\left(\frac{a_N}{(a_N+b_N)\log(a_N+b_N)}\right)^{1-\varepsilon}\right)^{b_N/a_N}$$

$$\leqslant\exp\{-b_N/a_N^{\varepsilon}((a_N+b_N)\log(a_N+b_N))^{1-\varepsilon}\}.$$

由条件(1.1.49)即得(1.1.68)式成立，这就是说我们有

$$\underline{\lim_{N\to\infty}}\max_{0\leqslant j\leqslant [b_N/a_N]}\beta(b_N+a_N, a_N)|W((j+1)a_N)-W(ja_N)|$$

$$\geqslant 1-\varepsilon \qquad \text{a.s.}$$

由于$a_T/a_{[T]}\to 1$，$b_T/b_{[T]}\to 1$，那么我们有

(1.1.69) $$\underline{\lim_{T\to\infty}}\max_{0\leqslant j\leqslant [b_{[T]}/a_{[T]}]}\beta(a_T+b_T, a_T)|W((j+1)a_{[T]})$$

$$-W(ja_{[T]})|\geqslant 1-\varepsilon \qquad \text{a.s.}$$

注意到我们有不等式

(1.1.70) $$\underline{\lim_{T\to\infty}}\sup_{0\leqslant t\leqslant b_T}\beta(t+a_T, a_T)|W(t+a_T)-W(t)|$$

$$\geqslant\underline{\lim_{T\to\infty}}\max_{0\leqslant j\leqslant [b_{[T]}/a_{[T]}]}\beta(a_T+b_T, a_T)|W((j+1)a_{[T]})$$

$$-W(ja_{[T]})|$$

$$
-\overline{\lim_{T\to\infty}}\sup_{0\leqslant t\leqslant b_T}\sup_{0\leqslant s\leqslant |a_T-a_{[T]}|}\beta(a_T+b_T,a_T)
\times|W(t+s)-W(t)|.
$$

显然地，上式右边第二项不超过$\varepsilon$. 故综合（1.1.69)、(1.1.70)就得

$$
\lim_{T\to\infty}\sup_{0\leqslant t\leqslant b_T}\beta(t+a_T,\ a_T)|W(t+a_T)-W(t)|\geqslant 1-2\varepsilon\quad \text{a.s.}
$$

这就得证（1.1.51）和（1.1.52）成立.定理1.1.4证毕.

**注1.1.3** 设$0<a_T\leqslant T$.从（1.1.47）我们有

$$
\text{(1.1.47}'\text{)}\quad \overline{\lim_{T\to\infty}}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\beta_T|W(t+s)-W(t)|
$$

$$
\leqslant\overline{\lim_{T\to\infty}}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}|W(t+s)-W(t)|/d(t+a_T,a_T)
$$

$$
\leqslant 1 \qquad \text{a.s.}
$$

另一方面，在（1.1.48）中取$b_T=T$，我们有

$$
\text{(1.1.48}'\text{)}\quad \overline{\lim_{T\to\infty}}\beta_T|W(T+a_T)-W(T)|
$$

$$
=\overline{\lim_{T\to\infty}}\ \beta(T+a_T,\ a_T)\ |W(T+a_T)-W(T)|=1\quad \text{a.s.}
$$

成立，只要

$$
\text{(1.1.71)}\qquad \underline{\lim_{T\to\infty}}\ a_T>0.
$$

综合(1.1.47′)和(1.1.48′)就得定理1.1.1的(1.1.1)—(1.1.4)，这就是说那里的条件（i）和（ii）可被条件（1.1.71）代替，而（1.1.71）是平凡的.

容易看出(1.1.47)蕴含如下的关于滞后增量的结果(Hanson and Russo, 1983a; Chen, Kong and Lin, 1986).若$a_T\to\infty$,那么

$$
\text{(1.1.72)}\quad \overline{\lim_{T\to\infty}}\sup_{0\leqslant t\leqslant T-a_T}|W(t+a_T)-W(t)|/d(t+a_T,a_T)\leqslant 1 \quad \text{a.s.}
$$

$$
\text{(1.1.73)}\quad \overline{\lim_{T\to\infty}}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}|W(t+s)-W(t)|/d(t+a_T,a_T)
\leqslant 1\quad \text{a.s.}
$$

若附设$a_T$是到上的，那么在(1.1.72)和(1.1.73)中等式成立．进一步，若$a_T$是$T$的连续函数且满足定理1.1.1的条件(iii)，那么在(1.1.72)和(1.1.73)式中，$\overline{\lim}$可被lim代替且成立等式．

若附设条件(1.1.71)及对任一$\varepsilon>0$

$$\sum_{N=1}^{\infty} \exp\left\{-\left(\frac{N\log N}{a_N}\right)^{\varepsilon} \Big/ \log N\right\}<\infty \tag{1.1.49'}$$

和

$$\lim_{T\to\infty} a_T/a_{[T]}=1, \tag{1.1.50'}$$

那么定理1.1.1的(1.1.5)和(1.1.6)也成立．

## §1.2 Wiener过程增量的某些下限结果

在§1.1中，我们讨论了Wiener过程的增量并获得某些上限结果，对Csörgő-Révész增量也给出了若干下限结果．在本节中，我们将对滞后增量详细地研究它的下限结果．

何凤霞和陈斌(1989)研究了定理1.1.3′的下限形式，证明了下述定理，它对应于Book和Shore(1978)关于Csörgő-Révész增量的结论．

**定理1.2.1**(何凤霞，陈斌，1989)　设$0<a_T\leqslant T$是$T$的不减函数且满足

(iv)　$\lim\limits_{T\to\infty}(\log T/a_T)/\log\log T=r\qquad 0\leqslant r\leqslant\infty$．

那么我们有

$$\underline{\lim_{T\to\infty}}\ \sup_{a_T\leqslant t\leqslant T}\ \sup_{t\leqslant s\leqslant T}|W(s)-W(s-t)|/d(T,t)=\alpha_r \quad \text{a.s.} \tag{1.2.1}$$

$$\underline{\lim_{T\to\infty}}\ \sup_{a_T\leqslant t\leqslant T}\ \sup_{t\leqslant s\leqslant T}\ \sup_{0\leqslant h\leqslant t}|W(s)-W(s-h)|/d(T,t)=\alpha_r \quad \text{a.s.} \tag{1.2.2}$$

其中　$\alpha_r=(r/(r+1))^{1/2}$．

**证**　首先，我们来证

$$(1.2.3)\qquad \varliminf_{T\to\infty}\ \sup_{a_T\leqslant t\leqslant T}\ \sup_{t\leqslant s\leqslant T}|W(s)-W(s-t)|/d(T,t)\geqslant\alpha_r\quad \text{a.s.}$$

对$r=0$情形，显然(1.2.3)式成立．对$0<r\leqslant\infty$情形，从（iv）即知条件（vi）被满足，因此由定理1.1.2我们有

$$\begin{aligned}(1.2.3)\text{式左边}&\geqslant\varliminf_{T\to\infty}\ \sup_{0\leqslant s\leqslant T-a_T}|W(s+a_T)-W(s)|/d(T,a_T)\\ &\geqslant\varliminf_{T\to\infty}\left(\frac{2a_T(\log T/a_T-\log\log\log T)}{2a_T(\log T/a_T+\log\log a_T)}\right)^{1/2}\\ &\geqslant\alpha_r\qquad \text{a.s.}\end{aligned}$$

现在为证明定理1.2.1，只需证明

$$(1.2.4)\qquad \varlimsup_{T\to\infty}\ \sup_{a_T\leqslant t\leqslant T}\ \sup_{t\leqslant s\leqslant T}\ \sup_{0\leqslant h\leqslant t}|W(s)-W(s-h)|/d(T,t)\leqslant\alpha_r\qquad \text{a.s.}$$

由(1.1.26″)可知(1.2.4)对$r=\infty$情形成立．我们仅需证明对$T_n=e^{\theta^n}$和$0\leqslant r<\infty$成立着

$$(1.2.5)\qquad \varlimsup_{n\to\infty}\ \sup_{a_{T_n}\leqslant t\leqslant T_n}\ \sup_{t\leqslant s\leqslant T_n}\ \sup_{0\leqslant h\leqslant t}|W(s)-W(s-h)|/d(T_n,t)\leqslant\alpha_r\qquad \text{a.s.}$$

对任$-\alpha_r<\alpha<1$，我们取实数$\theta>1$，$\varepsilon>0$使得

$$2\alpha^2/(2+\varepsilon)\theta>r/(r+1)+\varepsilon.$$

令$K_n=[\log_\theta T_n/a_{T_n}]$，$t_k=\theta^k a_{T_n}$，$0\leqslant k\leqslant K_n$．我们有

$$\begin{aligned}&\sup_{a_{T_n}\leqslant t\leqslant T_n}\ \sup_{t\leqslant s\leqslant T_n}\ \sup_{0\leqslant h\leqslant t}|W(s)-W(s-h)|/d(T_n,t)\\ &\leqslant\max_{0\leqslant k\leqslant K_n}\ \sup_{t_k\leqslant t\leqslant t_{k+1}}\ \sup_{t\leqslant s\leqslant T_n}\ \sup_{0\leqslant h\leqslant t}|W(s)-W(s-h)|/d(T_n,t_k)\\ &\leqslant\max_{0\leqslant k\leqslant K_n}\ \sup_{0\leqslant s-h,s\leqslant T_n}\ \sup_{0\leqslant h\leqslant t_{k+1}}|W(s)-W(s-h)|/d(T_n,t_k)\\ &=:\max_{0\leqslant k\leqslant K_n}A_{nk}.\end{aligned}$$

从条件（iv）及(1.1.31)式，当$n$充分大时，我们有

$$P\{A_{nk}\geqslant\alpha\}\leqslant c\ \frac{T_n}{t_{k+1}}\exp\left\{-\frac{2\alpha^2}{(2+\varepsilon)\theta}\left(\log\frac{T_n}{t_k}+\log\log t_k\right)\right\}$$

$$\leqslant c(\log T_n)^{(r+\varepsilon)-(r+\varepsilon+1)\frac{2\alpha^2}{(2+\varepsilon)\theta}}\theta^{-k(1-\alpha^2)}$$
$$\leqslant ce^{-n\varepsilon^2}\theta^{-k(1-\alpha^2)}.$$

所以

$$\sum_{n=1}^{\infty}P\{\max_{0\leqslant k\leqslant K_n}A_{nk}\geqslant a\}<\infty.$$

由Borel-Cantelli引理得证(1.2.5)成立。定理证毕。

Hanson和Russo (1989) 还讨论了 Wiener 过程滞后增量另一类型的下限结果，其中之一是下述定理。

**定理1.2.2** (Hanson and Russo, 1989) 设$0<a_T\leqslant T$，那么

(1.2.6) $$\varliminf_{T\to\infty}\sup_{0<t\leqslant T}\sup_{0\leqslant s\leqslant t}|W(T)-W(T-s)|/d(T,t)=0 \quad \text{a.s.}$$

(1.2.7) $$\varliminf_{T\to\infty}\sup_{a_T\leqslant t\leqslant T}(W(T)-W(T-t))/d(T,t)$$
$$\begin{cases}\text{在}[-1,g(a)]\text{中} & \text{若}\varliminf\limits_{T\to\infty}a_T/T=a,\\ =g(a) & \text{若}\lim\limits_{T\to\infty}a_T/T=a,\end{cases}$$

其中

$$g(a)=\begin{cases}0 & \text{若}a=0,\\ -1/(1-(\log a)/4)^{1/2} & \text{若}0<a\leqslant 1.\end{cases}$$

在定理的证明中我们将用到下面两引理。

**引理1.2.1** (Strassen, 1964) 定义

$$\eta_T(x)=W(Tx)/(2T\log\log T)^{1/2} \qquad 0\leqslant x\leqslant 1.$$

那么在空间$C[0,1]$中，序列$\{\eta_T(x)\}$概率为1地相对紧且极限点集是$K$。其中$K$是满足$f(0)=0$且$\int_0^1(f'(x))^2dx\leqslant 1$的绝对连续函数$f(x)$的全体。

引理的证明可在Csörgő和Révész(1981)的定理1.3.2中找到。

**引理1.2.2** 假设$a<b$且在$x=a$及$x=b$时$f(x)=\alpha x+\beta$，又设$f(x)$在$[a,b]$上绝对连续，其Radon-Nikodym导数为$f'(x)$。那么

$$\int_a^b (f'(x))^2 dx \geqslant \int_a^b \alpha^2 dx$$

且等式成立当且仅当对一切$x\in[a,b]$，$f(x)=\alpha x+\beta$.

**证** 设$\mu$是Lebesgue测度，$P=\mu/(b-a)$. 那么$X=f'$是概率空间（$[a,b]$，$\Sigma$，$P$）上的一个随机变量，其中$\Sigma$是$[a,b]$的一切$\mu$-可测子集全体. 此时

$$\frac{1}{b-a}\int_a^b (f'(x))^2 dx = EX^2 = E(X-EX)^2+(EX)^2$$

$$=E(X-\alpha)^2+\alpha^2 \geqslant \alpha^2 = \frac{1}{b-a}\int_a^b \alpha^2 dx$$

且不等式是严格的，除非$f'=X=\alpha$ a.s.

定理1.2.2的证明

1° 先让我们来证(1.2.6)式. 注意到对$0<t\leqslant T$一致地有

$$\lim_{T\to\infty}\beta^{-1}(T,t)/d(T,t)=1.$$

所以只需证明(1.2.6)中当$d(T,t)$ 被$\beta^{-1}(T,t)$ 代替时成立.

对任意给定的$\varepsilon>0$，$k\geqslant 1$和$-\infty<n\leqslant k$，定义

$$E_{nk}:=\{\sup_{0\leqslant t\leqslant 2^n}|W(2^k)-W(2^k-t)|\beta(2^k,2^n)>\sqrt{\varepsilon}\}$$

$$E_k=\bigcup_{-\infty<n\leqslant k}E_{nk}.$$

那么对一切充分大的$k$

$$P(E_{nk})\leqslant 4P\{W(2^n)>\sqrt{\varepsilon}\beta^{-1}(2^k,2^n)\}$$

$$\leqslant \exp\{-\varepsilon(\log 2^{k-n}+\log\log 2^k)\}\leqslant c2^{-\varepsilon(k-n)}k^{-\varepsilon},$$

由此可得

$$P(E_k)\leqslant c2^{-\varepsilon k}k^{-\varepsilon}\sum_{-\infty<n\leqslant k}(2^\varepsilon)^n=ck^{-\varepsilon}\to 0 \qquad (k\to\infty).$$

所以存在一个子列$\{k'\}$使得$P(\overline{\lim_{k'}}E_{k'})=0$.若$\omega\in(\overline{\lim_{k'}}E_{k'})^c$，那么对一切充分大$k'$（如$k'\geqslant K_0$）有$\omega\in E_{k'}^c$. 假设$k'>K_0$且$0<t\leqslant 2^{k'}$，那么对某$-\infty<n<k'$，有$2^{n-1}\leqslant t\leqslant 2^n$. 所以若$0\leqslant s\leqslant t$，由于$\omega$不在$E_{nk'}$中，我们有

$$|W(2^{k'})-W(2^{k'}-s)|\beta(2^{k'},\ t)\leqslant \sup_{0\leqslant s\leqslant 2^n}|W(2^{k'})$$

$$-W(2^{k'}-s)|\beta(2^{k'},\ 2^n)\sqrt{2}\leqslant\sqrt{2\varepsilon}.$$

这样，(1.2.6)中的$\overline{\lim}$小于或等于$\sqrt{2\varepsilon}$，由$\varepsilon>0$的任意性就证明了 (1.2.6) 式。

2° 我们来证当$a_T/T\to a>0$时有

$$(1.2.8)\quad \varlimsup_{T\to\infty}\sup_{a_T\leqslant t\leqslant T}(W(T)-W(T-t))/d(T,t)=g(a)\quad \text{a.s.}$$

因$a_T/T\to a>0$，那么当$T$充分大时，对$a_T\leqslant t\leqslant T$，$\log T/t$关于$t$是一致有界的，这样我们可用 $(2t\log\log t)^{1/2}$代替$d(T,t)$。由引理1.2.1存在$\Omega_0\subset\Omega$，$P(\Omega_0)=1$，使当$\omega\in\Omega_0$时，$\{\eta_T(x,\omega)\}$在$C[0,1]$中是相对紧的。固定$\Omega_0$中的$\omega_0$。假设$\{T_n\}$被选取为使$T_n\to\infty$且

$$\lim_{n\to\infty}\sup_{a_{T_n}\leqslant t\leqslant T_n}\frac{W(T_n)-W(T_n-t)}{(2t\log\log t)^{1/2}}$$

$$=\varlimsup_{T\to\infty}\sup_{a_T\leqslant t\leqslant T}\frac{W(T)-W(T-t)}{d(T,t)}.$$

从引理1.2.1存在$T_n$的一个子列（仍记为$T_n$）和$K$中元$f_0$使得对$x\in[0,1]$一致地有

$$W(T_nx)/\sqrt{2T_n\log\log T_n}-f_0(x)\to 0.$$

这样当$n\to\infty$时，关于$0\leqslant t\leqslant T_n$一致地有

$$\frac{W(T_n\cdot 1)-W(T_n(1-t/T_n))}{\sqrt{2T_n\log\log T_n}}-(f_0(1)-f(1-t/T_n))\to 0.$$

即得（对$\omega_0$）

$$(1.2.9)\quad \varlimsup_{T\to\infty}\sup_{a_T\leqslant t\leqslant T}(W(T)-W(T-t))/d(T,t)$$

$$=\lim_{n\to\infty}\sup_{a_{T_n}\leqslant t\leqslant T_n}\frac{W(T_n)-W(T_n-t)}{\sqrt{2T_n\log\log T_n}}\sqrt{\frac{T_n}{t}}$$

$$=\sup_{a\leqslant s\leqslant 1}\frac{f_0(1)-f_0(1-s)}{\sqrt{s}}$$

$$\geqslant \inf_{f\in K} \sup_{a\leqslant s\leqslant 1} \frac{f(1)-f(1-s)}{\sqrt{s}}.$$

运用同样的逼近过程，由引理1.2.1可知，若 $f^*\in K$，那么存在一个子序列$T_n\to\infty$使得

$$\varlimsup_{n\to\infty} \sup_{a_{T_n}\leqslant t\leqslant T_n} \frac{W(T_n)-W(T_n-t)}{d(T_n,t)}$$

$$=\sup_{a\leqslant s\leqslant 1} \frac{f^*(1)-f^*(1-s)}{\sqrt{s}}.$$

由于$K$是紧集，存在$f^*\in K$使得

$$\sup_{a\leqslant s\leqslant 1} \frac{f^*(1)-f^*(1-s)}{\sqrt{s}}=\inf_{f\in K} \sup_{a\leqslant s\leqslant 1} \frac{f(1)-f(1-s)}{\sqrt{s}}.$$

它仅与 $a$ 有关，是 $a$ 的一个函数$g(a)$，其形式待下面确定. 由对称性

$$g(a)=-\sup_{f\in K} \inf_{a\leqslant s\leqslant 1} (f(1)-f(1-s))/\sqrt{s}$$

$$=-\sup_{f\in K} \inf_{a\leqslant s\leqslant 1} f(s)/\sqrt{s}.$$

假设 $h$ 是$K$中达到上确界的函数，即

(1.2.10) $\quad -g(a)=\sup\limits_{f\in K} \inf\limits_{a\leqslant s\leqslant 1} f(s)/\sqrt{s}=\inf\limits_{a\leqslant s\leqslant 1} h(s)/\sqrt{s}.$

取$f(s)=s$，就得$-g(a)>0$. 从（1.2.10）我们有

(1.2.11) $\quad h(s)/\sqrt{s}\geqslant -g(a) \qquad a\leqslant s\leqslant 1,$

而且至少存在一点$S\in[a,1]$使得等式成立.

现在我们来证

(1.2.12) $\quad h(s)\begin{cases}\text{是线性的} & \text{当}0\leqslant s\leqslant a,\\ =-g(a)\sqrt{s} & \text{当}a\leqslant s\leqslant 1.\end{cases}$

如若不然，假设在$[0,a]$中$h(s)$不是线性的. 定义

$$h_1(s)=\begin{cases}sh(a)/a & \text{当}0\leqslant s\leqslant a,\\ h(s) & \text{当}a\leqslant s\leqslant 1.\end{cases}$$

那么 $\inf\limits_{a\leqslant s\leqslant 1} h_1(s)/\sqrt{s}=\inf\limits_{a\leqslant s\leqslant 1} h(s)/\sqrt{s}$，且从引理1.2.2有

$$\int_0^1 (h_1'(s))^2 ds<\int_0^1 (h'(s))^2 ds\leqslant 1.$$

这样函数$h_2(s)=h_1(s)/\left\{\int_0^1(h_1'(s))^2ds\right\}^{1/2}$属于 $K$。但是 $\inf\limits_{a\leqslant s\leqslant 1} h_2(s)/\sqrt{s}>-g(a)$，这与(1.2.10)矛盾．其次，假设存在一个$s^*\in[a,\ 1]$使得$h(s^*)\neq -g(a)\sqrt{s^*}$．由(1.2.11)，$h(s^*)>-g(a)\cdot\sqrt{s^*}$．令$h_1(s)$是$y=-g(a)\sqrt{s}$在$s=s^*$的切线并设$h_2=\min\{h,h_1\}$．函数 $h_2$ 满足$h_2(s)/\sqrt{s}\geqslant -g(a)$ $(a\leqslant s\leqslant 1)$．再由引理1.2.2有

$$\int_0^1(h_2'(s))^2ds<\int_0^1(h'(s))^2ds\leqslant 1.$$

那么函数$h_3(s)=h_2(s)/\left\{\int_0^1(h_2'(s))^2ds\right\}^{1/2}$属于$K$且

$\inf\limits_{a\leqslant s\leqslant 1} h_3(s)/\sqrt{s}>-g(a)$将给出矛盾。

由定义，我们可认为

$$\int_0^1(h'(s))^2ds=1, \tag{1.2.13}$$

因若不然，可对$h(s)$乘以适当常数使之成立．这样从 (1.2.12) 我们有

$$h(s)=\begin{cases}-g(a)s/\sqrt{a} & \text{当}0\leqslant s\leqslant a,\\ -g(a)\sqrt{s} & \text{当}a\leqslant s\leqslant 1.\end{cases}$$

所以

$$h'(s)=\begin{cases}-g(a)/\sqrt{a} & \text{当}0\leqslant s\leqslant a,\\ -g(a)/(2\sqrt{s}) & \text{当}a\leqslant s\leqslant 1.\end{cases} \tag{1.2.14}$$

从(1.2.13)和 (1.2.14) 就解得

$$g(a)=-(1-(\log a)/4)^{-1/2}.$$

3° 若$a_T/T\to 0$，那么对每一$a>0$有

$$\varliminf_{T\to\infty}\sup_{a_T\leqslant t\leqslant T}(W(T)-W(T-t))/d(T,t) \tag{1.2.15}$$

$$\geqslant\varliminf_{T\to\infty}\sup_{aT\leqslant t\leqslant T}(W(T)-W(T-t))/d(T,t)=g(a)\text{a.s.}$$

所以(1.2.15)有下界$g(0)=0$。另一方面，从(1.2.6)它有上界0．

4° 我们有

$$(1.2.16)\quad -1=g(1)$$

$$=\varlimsup_{T\to\infty}\sup_{T\leqslant t\leqslant T}(W(T)-W(T-t))/d(T,t)$$

$$\leqslant\varlimsup_{T\to\infty}\sup_{a_T\leqslant t\leqslant T}(W(T)-W(T-t))/d(T,t)\quad \text{a.s.}$$

若$\varlimsup\limits_{T\to\infty}a_T/T=a=0$，那么从(1.2.6)可见(1.2.16)式右边有上界0。若$\varlimsup\limits_{T\to\infty}a_T/T=a>0$，那么对每一$0<\varepsilon<a$，我们有

(1.2.16) 式的右边

$$\leqslant\varlimsup_{T\to\infty}\sup_{(a-\varepsilon)T\leqslant t\leqslant T}(W(T)-W(T-t))/d(T,t)$$

$$=g(a-\varepsilon).$$

令$\varepsilon\to 0$，得证 (1.2.7) 成立。定理证毕。

**注1.2.1** 与Wiener过程滞后增量有关的另一些问题也曾被研究过，例如$\beta_T(W(T,\omega)-W(T-a_T,\omega))$和$\sup\limits_{a_T\leqslant t\leqslant T}|W(T,\omega)-W(T-t,\omega)|/d(T,t)$的极限点集在Hanson 和 Russo(1983a,b)，陈桂景、孔繁超和林正炎 (1986) 及刘坤会 (1985) 等文中都有讨论。

## § 1.3 Wiener过程增量的进一步讨论

自Wiener过程的增量问题提出之后，近 10 年来，某些增量的精确收敛速度已被若干作者所研究。

### 1.3.1 Wiener过程增量的精确收敛速度

设$\{X(t);t\geqslant 0\}$是定义在概率空间$(\Omega,\mathscr{F},P)$上的随机过程。我们引入下述定义。

**定义1.3.1** 称函数$a_1(t)(t\geqslant 0)$属于过程$X(t)$的上上类，记作$a_1\in UUC(X)$，若对几乎所有$\omega\in\Omega$，存在$t_0=t_0(\omega)$使对每一$t>t_0$有$X(t)<a_1(t)$。

**定义1.3.2** 称函数 $a_2(t)(t\geqslant 0)$ 属于过程 $X(t)$ 的上下类，记作 $a_2\in ULC(X)$，若对几乎所有 $\omega\in\Omega$，存在数列 $0<t_1=t_1(\omega)<t_2=t_2(\omega)<\cdots$，$t_i\to\infty(i\to\infty)$，使得 $X(t_i)\geqslant a_2(t_i)$，$i=1,2,\cdots$.

**定义1.3.3** 称函数 $a_3(t)(t\geqslant 0)$ 属于过程 $X(t)$ 的下上类，记作 $a_3\in LUC(X)$，若对几乎所有 $\omega\in\Omega$，存在数列 $0<t_1=t_1(\omega)<t_2=t_2(\omega)<\cdots$，$t_i\to\infty(i\to\infty)$，使得 $X(t_i)\leqslant a_3(t_i)$，$i=1,2,\cdots$.

**定义1.3.4** 称函数 $a_4(t)(t\geqslant 0)$ 属于过程 $X(t)$ 的下下类，记作 $a_4\in LLC(X)$，若对几乎所有 $\omega\in\Omega$，存在 $t_0=t_0(\omega)$ 使对每一 $t>t_0$ 有 $X(t)>a_4(t)$.

设 $0<a_T\leqslant T$ 是 $T$ 的非降函数，记

$$Y_1(T)=a_T^{-1/2}\sup_{0\leqslant t\leqslant T-a_T}(W(t+a_T)-W(t)),$$

$$Y_2(T)=a_T^{-1/2}\sup_{0\leqslant t\leqslant T-a_T}|W(t+a_T)-W(t)|,$$

$$Y_3(T)=a_T^{-1/2}\sup_{0\leqslant t\leqslant T-a_T}\ \sup_{0\leqslant s\leqslant a_T}(W(t+s)-W(t)),$$

$$Y_4(T)=a_T^{-1/2}\sup_{0\leqslant t\leqslant T-a_T}\ \sup_{0\leqslant s\leqslant a_T}|W(t+s)-W(t)|.$$

在§1.1中，我们已介绍过Wiener过程的增量有多大，在此我们打算进一步研究四个过程类 $Y_i(t)$，$i=1,2,3,4$，给出关于Wiener过程的四个增量的较精确的结果.

**定理1.3.1** (Révész, 1982) 设 $0<a_T\leqslant T$ 是 $T$ 的函数满足：

(i) $a_T$ 是不减的；

(ii) $T/a_T$ 是不减的；

(iii) $\lim\limits_{T\to\infty}(\log Ta_T^{-1})/\log\log T=\infty$，

且记

$$a_1(T)=a_1(T,\varepsilon)=(2\log Ta_T^{-1}+2\log\log T+$$
$$(3+\varepsilon)\log\log Ta_T^{-1}+(2+\varepsilon)\log\log\log T)^{1/2},$$

$$a_2(T)=(2\log Ta_T^{-1}+2\log\log T+$$
$$\log\log Ta_T^{-1}+2\log\log\log T)^{1/2},$$

$$a_3(T)=a_3(T,\varepsilon)=\Big(2\log Ta_T^{-1}+\log\log Ta_T^{-1}$$

$$-2\log\log\log T+\log\left(\frac{51^2}{\pi}+\varepsilon\right)\Big)^{1/2},$$

$$a_4(T)=a_4(T,\varepsilon)=(2\log Ta_T^{-1}+\log\log Ta_T^{-1}$$
$$-2\log\log\log T-\log(\pi(1+\varepsilon)))^{1/2}.$$

那么对任一$\varepsilon>0$和$i=1,2,3,4$我们有

(1.3.1) $$a_1(t)\in UUC(Y_i),$$

(1.3.2) $$a_2(t)\in ULC(Y_i),$$

(1.3.3) $$a_3(t)\in LUC(Y_i),$$

(1.3.4) $$a_4(t)\in LLC(Y_i).$$

**证** 1° (1.3.1) 的证明。由于

$$Y_1(T)=\min(Y_1(T),Y_2(T),Y_3(T),Y_4(T))$$
$$\leqslant\max(Y_1(t),Y_2(t),Y_3(t),Y_4(t))=Y_4(T),$$

因此只需对$i=4$给出证明。记

$$P(T,\varepsilon)=P\{Y_4(T)\geqslant a_1(T,\varepsilon)\}.$$

那么由引理1.1.4，我们有

(1.3.5) $$P(T,\varepsilon/2)=O((\log T)^{-1}(\log Ta_T^{-1})^{-1-\varepsilon/4}(\log\log T)^{-1-\varepsilon/4})$$
$$=O((\log T)^{-1}(w(T))^{-1-\varepsilon/4}(\log\log T)^{-2-\varepsilon/2}),$$

其中$w(T)=(\log Ta_T^{-1})/\log\log T$。现在设$T_k$是满足

(1.3.6) $$(\log T_k)(w(T_k))^{1+\varepsilon/4}(\log\log T_k)=k$$

的最小实数。那么由平凡的不等式$\log w(T)<\log\log T$，我们有

(1.3.7) $$P(T_k,\varepsilon/2)=O(k^{-1}(\log\log T_k)^{-1-\varepsilon/2})$$
$$=O(k^{-1}(\log k)^{-1-\varepsilon/2}).$$

由Borel-Cantelli引理即得

$$P\{Y_4(T_k)\geqslant a_1(T_k,\varepsilon/2)\text{ i.o.}\}=0.$$

从过程$a_T^{1/2}Y_4(T)$是不减的和不等式

$$a_{T_{k+1}}a_1(T_{k+1},\varepsilon/2)\leqslant a_{T_k}a_1(T_k,\varepsilon),$$

即得(1.3.1)式，在上式证明中用到下述平凡的关系式：

$$a_{T_{k+1}}/a_{T_k}\leqslant T_{k+1}/T_k=O((\log\log T_k)^{-1}(w(T_k))^{-1-\varepsilon/4})+1$$

$$\log\frac{T_{k+1}}{a_{T_{k+1}}}\leqslant\log\frac{T_k}{a_{T_k}}+\log\frac{T_{k+1}}{T_k}=w(T_k)\log\log T_k$$

$$+O((\log\log T_k)^{-1}(w(T_k))^{-1-\varepsilon/4}).$$

2° (1.3.2)的证明。只需对$i=1$给出证明。事实上，下述较强的结果将被证明：

$$P\{A_k \quad \text{i.o.}\}=P\{\sup_{T_k\leqslant s\leqslant T_{k+1}-a_{T_{k+1}}} a_{T_{k+1}}^{-1/2}(W(s+a_{T_{k+1}})-W(s))\geqslant a_2(T_{k+1}) \quad \text{i.o.}\}=1,$$

其中$\{T_k\}$由(1.3.6)定义。从(1.3.6)式我们有 $T_{k+1}-T_k\geqslant a_{T_{k+1}}$。

由引理1.1.3我们有

$$P(A_k)=O\Big(\frac{T_{k+1}-T_k}{a_{T_{k+1}}}\Big(\log\frac{T_{k+1}}{a_{T_{k+1}}}\Big)^{1/2}\frac{a_{T_{k+1}}}{T_{k+1}}(\log T_{k+1})^{-\varepsilon}$$

$$\times\Big(\log\frac{T_{k+1}}{a_{T_{k+1}}}\Big)^{-1/2}(\log\log T_{k+1})^{-1}\Big)$$

$$=O((\log\log T_{k+1})^{-2}(w(T_{k+1}))^{-1-\varepsilon/4}(\log T_{k+1})^{-1})$$

$$=O((k\log\log T_{k+1})^{-1})=O((k\log k)^{-1}),$$

这就证明了(1.3.2)成立。

3° (1.3.3) 的证明。只需对 $i=4$ 给出证明。设 $T_k=\exp(k^{1+\rho}),k=1,2,\cdots,\quad \rho>0$并设

$$Z_4(k+1)=\sup_{T_k\leqslant t\leqslant T_{k+1}-a_{T_{k+1}}}\ \sup_{0\leqslant s\leqslant a_{T_{k+1}}} a_{T_{k+1}}^{-1/2}$$

$$\times|W(t+s)-W(t)|.$$

那么由引理1.1.4我们有

$$\sum_{k=1}^{\infty}P\{Z_4(k)<a_3(T_k)\}=\infty.$$

这就证明了

$$P\{Z_4(k)<a_3(t_k)\quad \text{i.o.}\}=1.$$

由于

$$Y_4(T_{k+1})\leqslant Z_4(T_{k+1})$$

$$+\sup_{0\leqslant t\leqslant T_k}\ \sup_{0\leqslant s\leqslant a_{T_{k+1}}} a_{T_{k+1}}^{-1/2}\ |W(t+s)-W(t)|,$$

又由 (1.3.1) 有

$$\sup_{0\leqslant t\leqslant T_k}\sup_{0\leqslant s\leqslant a_{T_{k+1}}}|W(t+s)-W(t)|=o(a_{T_{k+1}}^{-1/2}\,a_i(T_{k+1})),$$

这就证明了（1.3.3）成立。

$4^\circ$ （1.3.4)的证明。只需对$i=1$给出证明。由引理1.1.3当$T$充分大时有

$$P\{Y_1(T)\leqslant a_1(T,3\varepsilon)\}$$

$$\leqslant\exp\left\{-\frac{(1-\varepsilon)}{\sqrt{2\pi}}\,Ta_T^{-1}a_1(T)\exp\left(-\frac{a_1^2(T)}{2}\right)\right\}$$

$$\leqslant\exp\left\{\frac{-(1-\varepsilon)}{\sqrt{\pi}}\sqrt{\pi(1+3\varepsilon)}\log\log T\right\}\leqslant(\log T)^{-1-\delta},$$

其中$\delta$是一适当的正数。令$T_k=\exp(k^{1+\rho})$，$k=1,2,\cdots$；$\rho>0$。那么

$$\sum_{k=1}^{\infty}P\{Y_1(T_k)\leqslant a_1(T_k)\}<\infty,$$

由Borel-Cantelli 引理即得$P\{Y_1(T_k)\leqslant a_1(T_k)\quad \text{i.o.}\}=0$。设$T_k\leqslant T<T_{k+1}$。那么

$$a_T^{1/2}Y_1(T)\geqslant\sup_{0\leqslant t\leqslant T_k-a_{T_k}}(W(t+a_T)-W(t))$$

$$\geqslant\sup_{0\leqslant t\leqslant T_k-a_{T_k}}(W(t+a_{T_k})-W(t))$$

$$-\sup_{0\leqslant t\leqslant T_k-a_{T_k}}\sup_{0\leqslant u\leqslant a_{T_{k+1}}-a_{T_k}}|W(t+u)-W(t)|.$$

现在从（1.3.1）即得（1.3.4），定理证毕。

**推论1.3.1** 设$a_T$如定理1.3.1，但其中的条件（iii）换作较强的条件

$$\text{(iii}'\text{)}\qquad \lim_{T\to\infty}\frac{(\log Ta_T^{-1})^{1/2}}{\log\log T}=\infty,$$

那么我们有

(1.3.8) $\quad\lim_{T\to\infty}(Y_i(T)-(2\log Ta_T^{-1})^{1/2})=0\quad$ a.s. $i=1,2,3,4$.

注1.3.1 若条件（iii′）不成立，那么(1.3.8)也不能成立。事实上，若

(1.3.9) $$\lim_{T\to\infty}(\log Ta_T^{-1})^{1/2}/\log\log T=r>0,$$

那么对$i=1,2,3,4$有

$$0=\underline{\lim}_{T\to\infty}(Y_i(T)-(2\log Ta_T^{-1})^{1/2})$$

$$<\overline{\lim}_{T\to\infty}(Y_i(T)-(2\log Ta_T^{-1})^{1/2})=\frac{1}{r\sqrt{2}}\quad \text{a.s.}$$

若(1.3.9)中$r=0$，但（iii）仍成立，那么

$$\overline{\lim}_{T\to\infty}(Y_i(T)-(2\log Ta_T^{-1})^{1/2})=\infty\quad \text{a.s.}\quad i=1,2,3,4.$$

然而我们还是成立着

$$\lim_{T\to\infty}((2\log Ta_T^{-1})^{-1/2}Y_i(T)-1)=0\quad \text{a.s.}\quad i=1,2,3,4.$$

（见定理1.1.1）

### 1.3.2 下极限的速度有多快？

从(1.2.6)容易看出，对于$0<a_T\leqslant T$我们有

(1.3.10) $$\underline{\lim}_{T\to\infty}\sup_{a_T\leqslant t\leqslant T}|W(T)-W(T-t)|/d(T,t)=0\quad \text{a.s.}$$

Hanson 和 Russo（1989）曾提出这样的问题：什么样的因子将被用于来获得正的有限的下极限？这个问题的部分回答被刘坤会（1987）所给出。

**定理1.3.2**（刘坤会，1987） 设$0<a_T\leqslant T$是$T$的函数且满足

（iv） $\lim_{T\to\infty}(\log T/a_T)/\log\log T=r\qquad 0\leqslant r\leqslant\infty.$

那么存在着常数$C_1,C_2>0$使得

(1.3.11) $$C_1\sqrt{\frac{r}{1+r}}\leqslant I_i\leqslant C_2\sqrt{\frac{r}{1+r}}\qquad \text{a.s.}$$

其中

$$I_i=\underline{\lim}_{T\to\infty}\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}|W(T)-W(T-t)|/d(T,t),$$

$$I_2=\varlimsup_{T\to\infty}\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\sup_{t\leqslant s\leqslant t}|W(T)$$

$$-W(T-s)|/d(T,t),$$

$$I_3=\varlimsup_{T\to\infty}\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\beta(T,t)|W(T)-W(T-t)|,$$

$$I_4=\varlimsup_{T\to\infty}\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant t}\beta(T,t)$$

$$\times|W(T)-W(T-s)|,$$

我们可取 $C_1=\pi\sqrt{e-1}/(12\sqrt{e})$，$C_2=2e\sqrt{3.7/2.7}$。

定理1.3.2的证明将通过一系列引理来给出。

**引理1.3.1** 对$i=1,2,3,4$有

$$I_i\leqslant 2e \qquad \text{a.s.} \tag{1.3.12}$$

**证** 只需考察$I_2$。

1° 注意到当$T\geqslant e^e$时，$d(T,t)=\{2t(\log T/t+\log\log t)\}^{-1/2}$是$t(\leqslant T)$的增函数，所以为证（1.3.12）我们只需证明

$$\varlimsup_{T\to\infty}\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant t}|W(T)-W(T-s)|/d(T,t) \tag{1.3.13}$$

$$\leqslant\varlimsup_{T\to\infty}\sqrt{\log\log T}\sup_{0<t\leqslant T}|W(T)-W(T-t)|/d(T,t)\leqslant 2e \qquad \text{a.s.}$$

令$T_n=\exp(n^2)$，并注意到

$$\sup_{0<t\leqslant T_n}|W(T_n)-W(T_n-t)|/d(T_n,t)$$

$$\leqslant\sup_{0<t\leqslant T_n-T_{n-1}}\frac{|W(T_n)-W(T_n-t)|}{d(T_n,t)}$$

$$+\sup_{0\leqslant s\leqslant T_{n-1}}\frac{|W(T_{n-1})-W(T_{n-1}-s)|}{\{2(T_n-T_{n-1})\log\log(T_n-T_{n-1})\}^{1/2}}.$$

从定理1.1.3我们有

$$\varlimsup_{n\to\infty}\sqrt{\log\log T_n}\sup_{0\leqslant s\leqslant T_{n-1}}|W(T_{n-1})-W(T_{n-1}-s)| \tag{1.3.14}$$

$$/\{2(T_n-T_{n-1})\log\log(T_n-T_{n-1})\}^{1/2}=0 \qquad \text{a.s.}$$

若我们可以证明

$$(1.3.15)\quad \varlimsup_{n\to\infty}\sqrt{\log\log T_n}\sup_{0\leqslant t\leqslant T_n-T_{n-1}}|W(T_n)-W(T_n-t)|/d(T_n,t)\leqslant 2e \quad \text{a.s.}$$

那么从（1.3.14）和（1.3.15）式即得（1.3.13）成立.

2° 为证(1.3.15)式，设$T=e^n$.首先我们来证对任一$a\geqslant 2e$，存在一个正的常数$N=N(a)$使对每一$n\geqslant N$有

$$(1.3.16)\quad P\left\{\sqrt{\log\log T}\sup_{0<t\leqslant T}\frac{|W(T)-W(T-t)|}{d(T,t)}<a\right\}\geqslant n^{-3e^2/((e-1)a^2)}.$$

记$B(t)=W(T)-W(T-t)$.那么$\{B(t);0\leqslant t\leqslant T\}$是Wiener过程.由Wiener过程的增量独立性我们有

$$\begin{aligned}(1.3.17)\quad I_1&=P\{\sqrt{\log\log T}\sup_{0<t\leqslant T}|B(t)|/d(T,t)<a\}\\ &\geqslant P\{\sqrt{\log n}\sup_{-\infty<k\leqslant n-1}\sup_{e^k\leqslant t\leqslant e^{k+1}}|B(t)|/\\ &\qquad\{2e^k((n-k)+\log k)\}^{1/2}<a\}\\ &\geqslant\prod_{k=-\infty}^{n-1}P\{\sqrt{\log n}\sup_{e^k\leqslant t\leqslant e^{k+1}}|B(t)-B(e^k)|\\ &\leqslant(\{2e^k((n-k)+\log k)\}^{1/2}-\{2e^{k-1}((n-k+1)\\ &\qquad+\log(k-1))\}^{1/2})a\}.\end{aligned}$$

注意到对充分大$n$和任一$k$，我们有

$$\begin{aligned}&\{2e^k((n-k)+\log k)\}^{1/2}-\{2e^{k-1}((n-k+1)\\ &+\log(k-1))\}^{1/2}\\ &>\frac{\sqrt{e-1}}{2e}\{2e^k(e-1)((n-k)+\log k)\}^{1/2}\end{aligned}$$

和

$$\{\sup_{e^k\leqslant t\leqslant e^{k+1}}|B(t)-B(e^k)|\}\overset{\mathscr{D}}{=\!=}\{\sup_{0\leqslant s\leqslant e^k(e-1)}|B(s)|\}.$$

所以

$$(1.3.18)\quad I \geqslant \prod_{k=-\infty}^{n-1} P\{\sqrt{\log n} \sup_{0 \leqslant s \leqslant e^{k}(e-1)} |B(s)| < \frac{\sqrt{e-1}}{2e}$$

$$\times \{2e^{k}(e-1)((n-k)+\log k)\}^{1/2}a\} =: \prod_{k=-\infty}^{n-1} J_{nk}.$$

利用不等式 $2(1-\Phi(u)) \leqslant \exp(-u^2/2)$，$(u>0)$，若 $a>2e$，那么对一切 $-\infty<k\leqslant n-1$ 有

$$J_{nk} \geqslant 1 - 4\left(1-\Phi\left(\frac{\sqrt{e-1}}{2e}\{2((n-k)+\log k)/\log n\}^{1/2}a\right)\right)$$

$$\geqslant 1 - 2\exp\left\{-\frac{e-1}{4e^2}a^2((n-k)+\log k)/\log n\right\}$$

$$\geqslant \exp\left\{-2.6\exp\left\{-\frac{e-1}{4e^2}a^2((n-k)+\log k)/\log n\right\}\right\},$$

这样我们有

$$(1.3.19)\quad I \geqslant \exp\left\{-2.6\prod_{k=-\infty}^{n}\exp\left\{-\frac{e-1}{4e^2}a^2((n-k)\right.\right.$$

$$\left.\left.+\log k)/\log n\right\}\right\}.$$

注意到当 $n$ 充分大时，对任一 $a\geqslant 2e$ 关于 $n$ 一致地有

$$2.6\sum_{k=-\infty}^{n}\exp\left\{-\frac{e-1}{4e^2}a^2((n-k)+\log k)/\log n\right\}$$

$$\leqslant \frac{1}{2}\left(1-\exp\left\{-\frac{(e-1)a^2}{4e^2\log n}\right\}\right)^{-1},$$

所以

$$(1.3.20)\quad I \geqslant \exp\left\{-\frac{1}{2}\left(1-\exp\left\{-\frac{(e-1)a^2}{4e^2\log n}\right\}\right)^{-1}\right\}.$$

但对任一 $a\geqslant 2e$ 存在着数 $N=N(a)$ 使对每一 $n\geqslant N$ 有

$$1-\exp\{-(e-1)a^2/(4e^2\log n)\} \geqslant (e-1)a^2/(6e^2\log n).$$

把它代入(1.3.20)的右边，我们证得（1.3.16）式成立.

最后，令$a=2e$，从(1.3.16)我们有

$$P\{\sqrt{\log\log T_n}\sup_{0<t\leqslant T_n-T_{n-1}}|W(T_n)-W(T_n-t)|$$

$$/d(T_n,t)<2e\}$$

$$\geqslant P\{\sqrt{\log\log T_n}\sup_{0<t\leqslant T_n}|W(T_n)-W(T_n-t)|$$

$$/d(T_n,t)<2e\}$$

$$\geqslant n^{-3/(4(e-1))}.$$

由Borel-Cantelli引理得证(1.3.15)成立.引理1.3.1证毕.

**引理1.3.2** 假设对$0\leqslant r\leqslant 2.7$条件(iv)被满足，那么我们有

(1.3.21) $\qquad I_i\leqslant \pi e\sqrt{r}/2\sqrt{e-1}\quad$ a.s. $i=1,2,3,4$.

证 也只需考察$I_2$.

1° 设$0<a\leqslant 2e$，$T=e^n$.首先我们来证对任一$\varepsilon>0$存在着数$N=N(\varepsilon)$使得当$n\geqslant N$时我们有

(1.3.22) $$P\{\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant t}|W(T)-W(T-s)|$$

$$/d(T,t)<a\}$$

$$\geqslant n^{-(r+\varepsilon)\pi^2e^2/(4(e-1)a^2)}.$$

记$n_0=\min\{[\log a_T],n-1\}$，$B(t)=W(T)-W(T-t)$，$0\leqslant t\leqslant T$。那么仿照(1.3.16)的证明，我们有

(1.3.23) $$P\{\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant t}|W(T)-W(T-s)|$$

$$/d(T,t)<a\}$$

$$\geqslant P\left\{\sqrt{\log n}\sup_{n_0\leqslant k\leqslant n-1}\sup_{e^k\leqslant t\leqslant e^{k+1}}\sup_{0\leqslant s\leqslant t}\right.$$

$$\left.\times\frac{|B(s)|}{\{2e^k((n-k)+\log k)\}^{1/2}}<a\right\}$$

$$\geqslant P\{\sqrt{\log n}\sup_{0\leqslant s\leqslant e^{n_0}}|B(s)|$$

$$<a\{2e^{n_0-1}((n-n_0+1)+\log(n_0-1))\}^{1/2}\}$$

$$\times\prod_{k=n_0}^{n-}P\{\sqrt{\log n}\sup_{e^k\leqslant s\leqslant e^{k+1}}|B(s)-B(e^k)|$$

$$<a(\{2e^k((n-k)+\log k)\}^{1/2}-\{2e^{k-1}((n-k+1)$$
$$+\log(k-1))\}^{1/2}\}$$
$$\geqslant P\{\sqrt{\log n}\sup_{0\leqslant s\leqslant e^{n_0}}|B(s)|<a\{2e^{n_0-1}((n-n_0+1)$$
$$+\log(n_0-1))\}^{1/2}\}$$
$$\times\prod_{k=n_0}^{n-1}P\{\sqrt{\log n}\sup_{0\leqslant s\leqslant e^k(e-1)}|B(s)|$$
$$<a\frac{\sqrt{e-1}}{2e}\{2e^k(e-1)((n-k)+\log k)\}^{1/2}\}$$
$$\geqslant P\{\sup_{0\leqslant s\leqslant 1}|B(s)|<a\sqrt{2/e}\}(P\{\sup_{0\leqslant s\leqslant 1}|B(s)|$$
$$<e^{-1}a\sqrt{(e-1)/2}\})^{n-n_0}$$
$$\geqslant\exp\{-(n-n_0+1)\pi^2e^2/(4(e-1)a^2)\},$$
在后一不等式中，我们应用了熟知的不等式（例如见 Csörgo 和 Révész （1981） 的定理1.5.1）
$$P\{\sup_{0\leqslant s\leqslant 1}|B(s)|<u\}\geqslant\frac{4}{\pi}\left(e^{-\pi^2/8u^2}-\frac{1}{3}e^{-9\pi^2/8u^2}\right)$$
$$\geqslant\exp(-\pi^2/8u^2)\qquad(\text{当}u\leqslant 4.7).$$
从假设可知，当$n$充分大时显然有 $(n-n_0+1)/\log n<r+\varepsilon$。所以对任一$\varepsilon>0$存在$N=N(\varepsilon)$使当$n\geqslant N$时有
$$P\{\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\sup_{t\leqslant s\leqslant t}|W(T)-W(T-s)|/$$
$$d(T,t)<a\}$$
$$\geqslant\exp\{-(r+\varepsilon)\pi^2e^2(\log n)/(4(e-1)a^2)\}$$
$$=n^{-(r+\varepsilon)\pi^2e^2/4(e-1)a^2},$$
即得证（1.3.22）成立。

2° 对任给的$\varepsilon>0$，记$T_n=\exp(n^{1+\varepsilon})$。仿照引理1.3.1的证明并利用（1.3.22），我们有
$$I_2\leqslant\pi e\sqrt{(1+\varepsilon)(r+\varepsilon)}/2\sqrt{e-1}\qquad\text{a.s.}$$
故由$\varepsilon$的任意性得（1.3.21）成立，引理1.3.2证毕。

**引理1.3.3** 假设对$0\leqslant r\leqslant\infty$条件（iv）被满足．那么当$0\leqslant r\leqslant 1$我们有

$$I_i\geqslant\pi\sqrt{r(e-1)}/(12\sqrt{e})\quad\text{a.s.}\quad i=1,2,3,4.$$

当$1\leqslant r\leqslant\infty$时，我们有

$$I_i\geqslant\pi\sqrt{e-1}/(12\sqrt{e})\quad\text{a.s.}\quad i=1,2,3,4.$$

**证** 只需对$I_3$给出证明．

1° 设$0<\varepsilon<1$，$T_n=\exp(n^{(1+r+\varepsilon)^{-1}})$．当$0\leqslant r\leqslant 1$时，对充分大的$n$我们有

$$(1.3.24)\qquad \inf_{T_{n-1}\leqslant T\leqslant T_n}\sup_{a_T\leqslant t\leqslant T}\beta(T,t)|W(T)-W(T-t)|$$

$$\geqslant\sup_{M_n\leqslant t\leqslant T_n}\beta(T_n,s)|W(T_n)-W(T_n-s)|$$

$$-\sup_{T_{n-1}\leqslant T\leqslant T_n}|W(T_n)-W(T)|/(2a_n'\log\log T_{n-1})^{1/2}$$

$$=:A_n-B_n,$$

其中

$$M_n=\sup_{T_{n-1}\leqslant T\leqslant T_n}a_T+T_n-T_{n-1},\qquad a_n'=\inf_{T_{n-1}\leqslant T\leqslant T_n}a_T.$$

从(1.3.24)不难推得

$$(1.3.25)\qquad I_3=\varliminf_{T\to\infty}\sqrt{\log\log T}$$

$$\sup_{a_T\leqslant t\leqslant T}\beta(T,t)|W(T)-W(T-t)|$$

$$\geqslant\varliminf_{n\to\infty}\sqrt{\log\log T_{n-1}}\,A_n-\varlimsup_{n\to\infty}\sqrt{\log\log T_{n-1}}\,B_n.$$

注意到

$$\sqrt{\log\log T_{n-1}}\,B_n=(\{2(T_n-T_{n-1})(\log T_n/(T_n-T_{n-1})$$

$$+\log\log(T_n-T_{n-1}))\}/\{2a_n'\})^{1/2}$$

$$\times\sup_{0\leqslant s\leqslant T_n-T_{n-1}}|W(T_n)-W(T_n-s)|/d(T_n,T_n-T_{n-1}),$$

且当$n\to\infty$时$T_n-T_{n-1}\to\infty$．这样由定理1.1.3和

$$\lim_{n\to\infty}\{2(T_n-T_{n-1})(\log T_n/(T_n-T_{n-1})$$

$$+\log\log(T_n-T_{n-1}))\}/\{2a_n'\}=0,$$

我们有

$$\overline{\lim_{n\to\infty}}\sqrt{\log\log T_{n-1}}\,B_n=0 \qquad \text{a.s.}$$

如果我们能证明：

$$(1.3.26) \qquad \underline{\lim_{n\to\infty}}\sqrt{\log\log T_{n-1}}\,A_n$$

$$\geqslant\frac{\pi\sqrt{(e-1)(1-\varepsilon)r}}{4\sqrt{e(1+(1+\varepsilon)r)}\sqrt{(1-\varepsilon)r/4+(1+\varepsilon)(1+r+\varepsilon)}} \qquad \text{a.s.}$$

那么从 $\varepsilon$ 的任意性和 $0\leqslant r\leqslant 1$ 就得

$$(1.3.27) \qquad I_3\geqslant\pi\sqrt{r(e-1)}/(12\sqrt{e}) \qquad \text{a.s.}$$

若 $1\leqslant r\leqslant\infty$，记 $a_T'=\max\{a_T, T/\log T\}$。从(1.3.27)并注意到 $\lim\limits_{T\to\infty}(\log T/a_T')/\log\log T=1$，我们有

$$I_3\geqslant\underline{\lim_{T\to\infty}}\sqrt{\log\log T}\sup_{a_T'\leqslant t\leqslant T}\beta(T,t)|W(T)-W(T-t)|$$

$$\geqslant\pi\sqrt{e-1}/(12\sqrt{e}) \qquad \text{a.s.}$$

2° 为证(1.3.26)式，设 $T=e^{n'}$（这里 $n'$ 不必是整数）。首先我们来证对任给的 $0<\varepsilon<1$ 存在 $N=N(\varepsilon)$，使对每一 $n'\geqslant N$，任一 $a>0$ 和 $0\leqslant r<\infty$ 有

$$(1.3.28) \quad P\{\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\beta(T,t)|W(T)-W(T-t)|<a\}$$

$$\leqslant n'^{-(1/4-\varepsilon^2(e-1)/16r(1+(1+\varepsilon)r)a^2)(1-\varepsilon)r}.$$

显然(1.3.28)对 $r=0$ 成立。现在我们可设 $r>0$。设 $\bar{n}=[n']$，$n_0=[\log a_T]$，$B(t)=W(T)-W(T-t)$，$0\leqslant t\leqslant T$。当 $n'$ 充分大时，我们有 $\bar{n}\geqslant n_0+2$。仿照引理1.3.1的证明可以推得

$$(1.3.29) \quad I:=P\{\sqrt{\log\log T}$$

$$\times\sup_{a_T\leqslant t\leqslant T}\beta(T,t)|W(T)-W(T-t)|<a\}$$

$$\leqslant P\{\sqrt{\log n}\max_{n_0+2\leqslant k\leqslant n'}\sup_{e^{k-1}\leqslant t\leqslant e^k}|W(T)$$

$$-W(T-t)|/(2e^k(n'-k+\log n'))^{1/2}<a\}.$$

记

$$B_k=\{\sqrt{\log n'}\sup_{e^{k-1}\leqslant t\leqslant e^k}|W(T)-W(T-t)|/$$

$$(2e^k(n'-k+\log n'))^{1/2}<a\},$$

$$B'_{\bar n}=\{\sqrt{\log n'}\sup_{0\leqslant t\leqslant e^{\bar n}(1-1/e)}|B(t)|/(2e^{\bar n}(n'-\bar n+\log n'))^{1/2}$$

$$<a\}.$$

用$P(\cdot\|x)$记自$x$出发的Brown运动分布。从 Wiener 过程的Markov性，我们有

$$P\Big\{\bigcap_{k=n_0+2}^{\bar n}B_k\Big\}=E\Big(I\Big(\bigcap_{k=n_0+2}^{\bar n-1}B_k\Big)P(B'_{\bar n}\|B(e^{\bar n-1}))\Big)$$

$$\leqslant P\Big\{\bigcap_{k=n_0+2}^{\bar n-1}B_k\Big\}P(B'_{\bar n}),$$

递推之，得到

(1.3.30)
$$I\leqslant\prod_{k=n_0+2}^{\bar n}P\Big\{\sqrt{\log n'}$$

$$\times\sup_{0\leqslant t\leqslant e^k(1-1/e)}\frac{|B(t)|}{\{2e^k[(n'-k)+\log n']\}^{1/2}}<a\Big\}$$

$$=\prod_{k=n_0+2}^{\bar n}P\Big\{\sup_{0\leqslant t\leqslant 1}|B(t)|<a\Big\{2\frac{e}{e-1}$$

$$\times[(n'-k)+\log n']/\log n'\Big\}^{1/2}\Big\}$$

$$\leqslant\Big(P\Big\{\sup_{0\leqslant t\leqslant 1}|B(t)|<a\Big\{2\frac{e}{e-1}$$

$$\times[(n'-n_0-2)+\log n']/\log n'\Big\}^{1/2}\Big\}\Big)^{\bar n-n_0-1}.$$

从引理的假设，容易看出对任给的$\varepsilon>0$存在$N=N(\varepsilon)$使对每一 $n'\geqslant N$

$$\bar n-n_0-1\geqslant(1-\varepsilon)r\log n',\quad (n'-n_0-2)/\log n'\leqslant(1+\varepsilon)r.$$

从Csörgö和Révész (1981) 的引理1.6.1，我们有

$$I \leqslant \left(P\left\{\sup_{0 \leqslant t \leqslant 1}|B(t)| < \alpha\sqrt{\frac{2e}{e-1}[1+(1+\varepsilon)r]}\right\}\right)^{(1-\varepsilon)r\log n'}$$

$$\leqslant \left(\frac{4}{\pi} e^{-\pi^2/8 \cdot \frac{2e}{e-1}\alpha^2[1+(1+\varepsilon)r]}\right)^{(1-\varepsilon)r\log n'}$$

$$\leqslant \exp(\{0.25 - \pi^2(e-1)/16e[1+(1+\varepsilon)r]\alpha^2\}(1-\varepsilon)r\log n')$$

$$= n'^{\{1/4-\pi^2(e-1)/16e(1+(1+\varepsilon)r)\alpha^2\}(1-\varepsilon)r}.$$

这就得证(1.3.28)成立。令$n=[n'^{1+r+\varepsilon}]$。现在从(1.3.28)我们有

$$\sum_{n=1}^{\infty} P\left\{\sqrt{\log\log T_{n-1}}\,A_n \leqslant \frac{\pi\sqrt{(e-1)(1-\varepsilon)r}}{4\sqrt{e(1+(1+\varepsilon)r)}\sqrt{(1-\varepsilon)r/4+(1+\varepsilon)(1+r+\varepsilon)}}\right\}$$

$$\leqslant \sum_{n=1}^{\infty} n^{-(1+\varepsilon)} < \infty.$$

从Borel-Cantelli引理得证（1.3.26）.引理1.3.3证毕.

**定理1.3.2的证明** 综合引理1.3.1,1.3.2和1.3.3即得定理1.3.2.

## § 1.4 两参数Wiener过程的增量有多大?

关于多参数Wiener过程，其对应增量的讨论通常是比较复杂的，然而关于Wiener过程增量的很多结果已被推广到两参数Wiener过程上.

### 1.4.1 Csörgő-Révész增量

设$\{W(s,t),\ 0\leqslant s,\ t<\infty\}$是两参数Wiener过程，即对于矩形$R=[x_1,\ x_2]\times[y_1,\ y_2]\in R_+^2(0\leqslant x_1< x_2<\infty,\ 0\leqslant y_1<y_2<\infty)$，$W$-测度

$$W(R)=W(x_2,y_2)-W(x_1,y_2)-W(x_2,y_1)+W(x_1,y_1)$$

满足下列性质：

（i） $W(R)\in N$（0，$\lambda(R)$），其中 $\lambda(R)=(x_2-x_1)(y_2-y_1)$；

（ii） $W(0,y)=W(x,0)=0 \qquad (0\leqslant x,y<\infty)$；

（iii） $\{W(s,t)\}$是一个独立增量过程，即当$R_1$，$R_2$，…，$R_n$是互不相交矩形时，$W(R_1)$，$W(R_2)$，…，$W(R_n)$是相互独立的随机变量；

（iv） 样本轨道$W(s,t;\ \omega)$概率为1地关于$s,t$连续。

设$0<a_T\leqslant T$，$b_T\geqslant T^{1/2}$是$T$的不减函数并记

（1.4.1）
$$\delta_T=\{2a_T(\log T/a_T+\log(\log b_T/a_T^{1/2}+1)+\log\log T)\}^{-1/2}.$$

又设$L_T=L_T(a_T,b_T)$（分别地$L_T^*=L_T^*(a_T,b_T)$）是矩形$R=[x_1,x_2]\times[y_1,y_2]\in D_T(b_T)$，$\lambda(R)\leqslant a_T$（分别地$\lambda(R)=a_T$）的集，其中

$$D_T(b_T)=\{(x,y):xy\leqslant T,\ 0\leqslant x\leqslant b_T,\ 0\leqslant y\leqslant b_T\}.$$

Csörgö和Révész(1978)首先讨论两参数Wiener过程的增量并证明了下述定理。

**定理1.4.1** (Csörgö and Révész, 1978) 假设

（i） $T/a_T$是$T$的不减函数；

（ii） $\delta_T$是$T$的不增函数；

（iii） 对任一$\varepsilon>0$存在$\theta_0=\theta_0(\varepsilon)>1$使当$1<\theta\leqslant\theta_0$时

（1.4.2）
$$\overline{\lim_{k\to\infty}}\,\delta_{\theta^k}/\delta_{\theta^{k+1}}\leqslant 1+\varepsilon.$$

那么

（1.4.3）
$$\overline{\lim_{T\to\infty}}\sup_{R\in L_T}\delta_T|W(R)|=\overline{\lim_{T\to\infty}}\sup_{R\in L_T^*}\delta_T|W(R)|=1 \quad \text{a.s.}$$

若还满足

（iv） $\lim\limits_{T\to\infty}(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1))/\log\log T=\infty$，

那么

(1.4.4) $$\lim_{T\to\infty}\sup_{R\in L_T}\delta_T|W(R)|=\lim_{T\to\infty}\sup_{R\in L_T^*}\delta_T|W(R)|=1$$ a.s.

关于$W(s,t)$ 的下述重对数律是定理1.4.1的一个推论.

**推论1.4.1** (Orey and Pruitt,1973; Park,1974等)我们有

(1.4.5) $$\overline{\lim_{T\to\infty}}\sup_{0\leqslant s,t\leqslant T,st=T}\frac{|W(s,t)|}{\sqrt{4T\log\log T}}=1 \qquad \text{a.s.}$$

林正炎 (1984) 首先讨论关于$\{W(R)\}$的下极限问题，证明：若条件 (iv) 不成立，那么在附加一较上面提到的条件来得弱的条件下，给出了对于$\underline{\lim}$ 的正则化因子，这是Csáki和Révész (1979) 的一个结果，即(1.1.10)，的一个类比.令

$$\lambda_T=\{2a_T(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1))\}^{-1/2}.$$

**定理1.4.2** (林正炎，1984) 假设

(i) $T/a_T$是$T$的不减函数;

(ii) $\lambda_T$是$T$的不增函数;

(iii) 对任一$\varepsilon>0$，存在$\theta_0=\theta_0(\varepsilon)>1$，使当$1<\theta\leqslant\theta_0$时
$$\overline{\lim_{k\to\infty}}\lambda_{\theta^k}/\lambda_{\theta^{k+1}}\leqslant1+\varepsilon;$$

(iv′) $$\lim_{T\to\infty}(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1))/\log\log\log T=\infty.$$

那么
$$\underline{\lim_{T\to\infty}}\sup_{R\in L_T}\lambda_T|W(R)|=\underline{\lim_{T\to\infty}}\sup_{R\in L_T^*}\lambda_T|W(R)|=1. \qquad \text{a.s.}$$

**证** 首先我们证明，存在正数序列$T_k\uparrow\infty$ $(k\to\infty)$ 使得

(1.4.6) $$\underline{\lim_{k\to\infty}}\sup_{R\in L_{T_k}^*}\lambda_{T_k}|W(R)|\geqslant1 \quad \text{a.s.}$$

先设$a_T/T\downarrow\rho<1$.考察区域$D_T$，在$[0,b_T]$中插入分点$0=x_0<x_1<\cdots<x_m\leqslant b_T(<x_{m+1})$ 使得

(1.4.7) $$(x_i-x_{i-1})y_i=a_T,$$

其中$y_i=b_T(0\leqslant i<i_0)$ 或者$T/x_i$ $(i_0\leqslant i\leqslant m+1)$，而
$$i_0:=\min\{i:T/x_i<b_T\}=[T/a_T]+1.$$

显然，$x'_{i_0}=T^2/(b_T(T-a_T))$ 是方程 $(x'_{i_0}-Tb_T^{-1})Tx'^{-1}_{i_0}=a_T$ 的解，且$x_{i_0}\leqslant x'_{i_0}$。利用(1.4.7)式，由归纳法易证

$$x_i\leqslant x'_i:=T^{i-i_0+2}/(b_T(T-a_T)^{i-i_0+1}),\quad i_0\leqslant i\leqslant m+1.$$

因$x'_{m+1}>b_T$，即$T^{m-i_0+3}/(b_T(T-a_T)^{m-i_0+2})>b_T$，我们有

$$\begin{aligned}(1.4.8)\qquad m&>i_0-2+(\log b_T^2T^{-1})/\log(T/(T-a_T))\\&=[Ta_T^{-1}]-1+(\log b_T^2T^{-1})/\log(T/(T-a_T)).\end{aligned}$$

置非负数集$I_1=\{T:b_Ta_T^{-1/2}<Ta_T^{-1}\}$，$I_2=\{T:b_Ta_T^{-1/2}\geqslant Ta_T^{-1}\}$。不妨设它们都是无界的。由条件(iv′)，

$$\lim_{T\to\infty,T\in I_1} Ta_T^{-1}=\infty$$

(事实上，利用(ii)，更有$\lim\limits_{T\to\infty}Ta_T^{-1}=\infty$)。进一步，当$T$沿着$I_1$趋于$\infty$时，还有$\log b_Ta_T^{-1/2}=o(Ta_T^{-1})$。因此$\lambda_T/(2a_T\log Ta_T^{-1})^{-1/2}\to1$。注意到 $m>[Ta_T^{-1}]-1$，由关于正态分布的尾概率不等式，对一切充分大的$T\in I_1$，我们有

$$\begin{aligned}(1.4.9)\qquad &P\{\sup_{R\in L_T^*}\lambda_T|W(R)|\leqslant1-\varepsilon\}\\
&\leqslant\left\{1-\frac{\lambda_T}{(1-\varepsilon)\sqrt{2\pi a_T}}\exp\left(-\frac{1}{2}(1-\varepsilon)^2a_T\lambda_T^{-2}\right)\right\}^m\\
&\leqslant\left\{1-\frac{1}{4(1-\varepsilon)\sqrt{\log Ta_T^{-1}}}\right.\\
&\qquad\left.\times\exp(-(1-2\varepsilon+2\varepsilon^2)\log Ta_T^{-1})\right\}^{[Ta_T^{-1}]-1}\\
&\leqslant\exp\left\{-\frac{[Ta_T^{-1}]-1}{4(1-\varepsilon)\sqrt{\log Ta_T^{-1}}}(Ta_T^{-1})^{-(1-2\varepsilon+2\varepsilon^2)}\right\}\\
&\leqslant\exp\{-(Ta_T^{-1})^\varepsilon\}.\end{aligned}$$

因为当$T$沿着$I_1$趋于$\infty$时，$(\log Ta_T^{-1})/\log\log\log T\to\infty$，所以对一切充分大的$T\in I_1$，$(\log Ta_T^{-1})\geqslant(2/\varepsilon)\log\log\log T$，即

$$(1.4.10)\qquad Ta_T^{-1}\geqslant(\log\log T)^{2/\varepsilon}$$

令$T_0=0$，$T_k=(1+k^{-1/2})^k$。易见当$k\to\infty$时$T_k\uparrow\infty$。因对适当大的$k$，

$$\log(1+k^{-1/2})\geqslant k^{-2/3},$$

所以只要$k$取得充分大，对$T_k\in I_1$就有

(1.4.11) $$(T_k a_{T_k}^{-1})^{\varepsilon}\geqslant(\log\log T_k)^2=\{\log(k\log(1+k^{-1/2}))\}^2$$
$$\geqslant(\log k^{1/3})^2\geqslant\log k^2.$$

将它代入（1.4.9）式右边的指数部分，即得

$$\sum_{k=1}^{\infty}{}' P\{\sup_{R\in L^*_{T_k}}\lambda_{T_k}|W(R)|\leqslant 1-\varepsilon\}<\infty,$$

其中$\Sigma'$表示对一切$T_k\in I_1$的$k$求和。由 Borel-Cantelli 引理和$\varepsilon$的任意性得证

(1.4.12) $$\varliminf_{k\to\infty,\ T_k\in I_1}\ \sup_{R\in L^*_{T_k}}\lambda_{T_k}|W(R)|\geqslant 1 \qquad \text{a.s.}$$

若$T\in I_2$，那么$b_T^2T^{-1}\geqslant b_Ta_T^{-1/2}$。分别讨论$\rho=0$和$0<\rho<1$两种情形，若$\rho=0$，由于对充分大的$T$，$\log\dfrac{T}{T-a_T}=\log\left(1+\dfrac{a_T}{T-a_T}\right)$

$\leqslant\dfrac{a_T}{T-a_T}$，所以我们有

$$\left(\log\frac{b_T^2}{T}\right)\Big/\left(\log\frac{T}{T-a_T}\right)\geqslant(Ta_T^{-1}-1)\log b_Ta_T^{-1/2}.$$

将它代入（1.4.8）式可得

(1.4.13) $$P\{\sup_{R\in L^*_T}\lambda_T|W(R)|\leqslant 1-\varepsilon\}$$
$$\leqslant\Big\{1-\frac{1}{\sqrt{2\pi}\,(1-\varepsilon)\sqrt{2(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1))}}$$
$$\times\exp(-(1-\varepsilon)^2(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)))\Big\}^m$$
$$\leqslant\exp\Big\{-\frac{[Ta_T^{-1}]-1+(Ta_T^{-1}-1)\log b_Ta_T^{-1/2}}{2(1-\varepsilon)\sqrt{\pi}(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1))}$$
$$\times(Ta_T^{-1}(\log b_Ta_T^{-1/2}+1))^{-(1-\varepsilon)^2}\Big\}$$
$$\leqslant\exp\{-(Ta_T^{-1}(\log b_Ta_T^{-1/2}+1))^{\varepsilon}\}.$$

类似于（1.4.12）式，我们有

(1.4.14) $$\varliminf_{k\to\infty,\ T_k\in I_2}\ \sup_{R\in L^*_{T_k}}\lambda_{T_k}|W(R)|\geqslant 1 \qquad \text{a.s.}$$

若$0<\rho<1$，那么对充分大的$T$，$\log\frac{T}{T-a_T}\leqslant\log\frac{2}{1-\rho}$。所以有

$$\left(\log\frac{b_T^2}{T}\right)\Big/\left(\log\frac{T}{T-a_T}\right)\geqslant\left(\log\frac{b_T}{\sqrt{a_T}}\right)\Big/\left(\log\frac{2}{1-\rho}\right).$$

从而类似于（1.4.13）式，有

(1.4.15) $$P\{\sup_{R\in L_T^*}\lambda_T|W(R)|\leqslant 1-\varepsilon\}\leqslant\exp\{-(\log b_T a_T^{-1/2})^{\varepsilon}\}.$$

注意到 $0<\rho<1$，由条件（iv′）我们有 $(\log\log(b_T a_T^{-1/2}))/\log\log\log T\to\infty$。所以(1.4.14)式也成立。结合(1.4.12)式得证当$\rho<1$时，（1.4.6）式成立。

下面来讨论$\rho=1$的情形。此时$a_T=T$（见Csörgő and Révész，(1981)）。由条件(iv′)，$(\log\log(b_T a_T^{-1/2}))/\log\log\log T\to\infty$。若记

$a_T'=(1-(\log\log T)^{-1})T$，$\lambda_T'=\{2a_T'\log(\log b_T a_T'^{-1/2}+1)\}^{-1/2}$，

设$L_T^{**}$是以$a_T'$代$a_T(=T)$后的$L_T^*$。则因

$$(\log b_T^2 T^{-1})\Big/\left(\log\frac{T}{T-a_T'}\right)=(\log\log\log T)^{-1}\log b_T^2 T^{-1},$$

类似于（1.4.13）式，对一切充分大的$T$

$$\begin{aligned}&P\{\sup_{R\in L_T^{**}}\lambda_T'|W(R)|\leqslant 1-\varepsilon\}\\&\leqslant\exp\left\{-\frac{-1+2(\log\log\log T)^{-1}\log b_T T^{-1/2}}{4(1-\varepsilon)\sqrt{\log(\log b_T a_T'^{-1/2}+1)}}\right.\\&\qquad\left.\times(\log b_T a_T'^{-\frac{1}{2}}+1)^{-(1-\varepsilon)^2}\right\}\\&\leqslant\exp\{-(\log b_T T^{-1/2})^{\varepsilon}\}.\end{aligned}$$

与（1.4.13）推出（1.4.14）式一样，我们也有

$$\varliminf_{k\to\infty}\sup_{R\in L_{T_k}^{**}}\lambda_{T_k}'|W(R)|\geqslant 1\quad\text{a.s.}$$

显然$\lambda_T'/\lambda_T\to 1(T\to\infty)$，因而我们有

(1.4.16) $$\varliminf_{k\to\infty}\sup_{R\in L_{T_k}^{**}}\lambda_{T_k}|W(R)|\geqslant 1\quad\text{a.s.}$$

又记

$a_T''=T/\log\log T$，$\lambda_T''=(2a_T''\log(\log b_T a_T''^{-1/2}+1))^{-1/2}$，

$\delta_T' = \{2a_T''(\log T a_T''^{-1} + \log(\log b_T a_T''^{-1/2} + 1) + \log\log T)\}^{-1/2}$，

$L_T'$是以$a_T''$代$a_T$后的$L_T$. 我们来证$T(\log\log T)^{-1}\log\log b_T T^{-1/2}$是不减的. 取充分大的常数$C$，对满足$T^{1/2}\log T \leqslant C$和 $T > e^e$ 的$T$，考察函数

$$f(T) = T(\log\log T)^{-1}\log\log CT^{-1/2}.$$

对上述$T$，易见有$f'(T) \geqslant 0$，即函数$f(T)$是不减的. 由条件(iv')对充分大的$T$有$T^{1/2}\log T < b_T$. 对任意给定的充分大的$T_1 < T_2$，取$C = b_{T_2}$，我们有

$$\frac{T_1}{\log\log T_1}\log\log b_{T_1}T_1^{-\frac{1}{2}} \leqslant \frac{T_1}{\log\log T_1}\log\log b_{T_2}T_1^{-\frac{1}{2}}$$

$$\leqslant \frac{T_2}{\log\log T_2}\log\log b_{T_2}T_2^{-\frac{1}{2}}.$$

由此即得$\delta_T'$是不增的. 进一步，由条件 (iii)，对任意的$\varepsilon > 0$，存在$\theta_0 = \theta_0(\varepsilon) > 1$使当$1 < \theta \leqslant \theta_0$时有

$$\overline{\lim_{k\to\infty}}\, \delta_{\theta^k}' / \delta_{\theta^{k+1}}' \leqslant 1 + \varepsilon.$$

容易看出$\lambda_T / \delta_T' \to 0\ (T \to \infty)$. 那么由定理1.4.1我们有

$$(1.4.17)\quad \overline{\lim_{T\to\infty}} \sup_{R\in L_T'} \lambda_T |W(R)| = \overline{\lim_{T\to\infty}} \sup_{R\in L_T'} \delta_T' |W(R)| \frac{\lambda_T}{\delta_T'} = 0 \quad \text{a.s.}$$

结合 (1.4.16) 式即得当$\rho = 1$时有

$$\underline{\lim_{k\to\infty}} \sup_{R\in L_{T_k}^*} \lambda_{T_k} |W(R)|$$
$$\geqslant \underline{\lim_{k\to\infty}} \sup_{R\in L_{T_k}^{**}} \lambda_{T_k} |W(R)| - \overline{\lim_{k\to\infty}}\, 4 \sup_{R\in L_{T_k}'} \lambda_{T_k} |W(R)| \geqslant 1 \quad \text{a.s.}$$

这就证明了对任意的$\rho \leqslant 1$，(1.4.6)式都成立.

其次，我们来证明：存在正整数子列$\{k_k\}$使得

$$(1.4.18)\quad \overline{\lim_{k\to\infty}} \sup_{R\in L_{T_{k_k}}} \lambda_{T_{k_k}} |W(R)| \leqslant 1 \quad \text{a.s.},$$

为此，令

$$D'_T=\{(x,y): xy\leqslant Ta_T^{-1},\ 0\leqslant x,y\leqslant b_Ta_T^{-1/2}\},$$
$$L''_T=\{R: R\subset D'_T,\ \lambda(R)\leqslant 1\},$$
$$K(T)=2(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1)).$$

由条件（iv′）和$b_T\geqslant T^{1/2}$我们有$\lim\limits_{k\to\infty}b_{T_k}a_{T_k}^{-1/2}=\infty$。由此可得存在正整数子列$\{k'\}$，使$b_{T_{k'}}a_{T_{k'}}^{-1/2}$单调不减趋于$\infty$。由Csörgő和Révész(1981)中的定理1.12.6可知，对任一$\varepsilon>0$存在$c_\varepsilon>0$使得

$$P\{\sup_{R\in L''_{T_{k'}}}(K(T_{k'}))^{-\frac{1}{2}}|W(R)|\geqslant 1+\varepsilon\}$$
$$\leqslant c_\varepsilon T_{k'}a_{T_{k'}}^{-1}(1+\log T_{k'}a_{T_{k'}}^{-1})(1+\log b_{T_{k'}}a_{T_{k'}}^{-1/2})$$
$$\times\exp\left\{-\frac{2(1+\varepsilon)^2}{2+\varepsilon}(\log T_{k'}a_{T_{k'}}^{-1}+\log(\log b_{T_{k'}}a_{T_{k'}}^{-1/2}+1))\right\}$$
$$\leqslant 2c_\varepsilon(T_{k'}a_{T_{k'}}^{-1}\log b_{T_{k'}}a_{T_{k'}}^{-1/2})^{-\varepsilon},$$

其中$T_{k'}a_{T_{k'}}^{-1}\log b_{T_{k'}}a_{T_{k'}}^{-1/2}\uparrow\infty(k'\to\infty)$。定义

$$k_n=\max\{k': T_{k'}a_{T_{k'}}^{-1}\log b_{T_{k'}}a_{T_{k'}}^{-1/2}\leqslant n^{2/\varepsilon}\}.$$

那么我们有

$$\overline{\lim_{n\to\infty}}\sup_{R\in L''_{T_{k_n}}}(K(T_{k_n}))^{-\frac{1}{2}}|W(R)|\leqslant 1+\varepsilon\quad \text{a.s.}$$

对$k_n<k'\leqslant k_{n+1}$，由于$L''_{T_{k'}}$是不减的，所以

$$\overline{\lim_{k'\to\infty}}\sup_{R\in L''_{T_{k'}}}(K(T_{k'}))^{-\frac{1}{2}}|W(R)|$$
$$\leqslant\overline{\lim_{n\to\infty}}\sup_{R\in L''_{T_{k_{n+1}}}}(K(T_{k_{n+1}}))^{-\frac{1}{2}}|W(R)|\left(\frac{K(T_{k_{n+1}})}{K(T_{k_n})}\right)^{\frac{1}{2}}$$
$$\leqslant 1+\varepsilon\quad \text{a.s.}$$

由$\varepsilon$的任意性得

$$\overline{\lim_{k'\to\infty}}\sup_{R\in L''_{T_{k'}}}(K(T_{k'}))^{-\frac{1}{2}}|W(R)|\leqslant 1\quad \text{a.s.}$$

而

$$\sup_{R\in L_{T_{k'}}}(K(T_{k'}))^{-\frac{1}{2}}|W(R)|\text{与}\sup_{R\in L_{T_{k'}}}\lambda_{T_{k'}}|W(R)|\text{ 同分布，}$$

所以对后者有 $\{k'\}$ 的子列 $\{k_k\}$，使得 (1.4.18) 式成立.结合(1.4.6)式即得

$$\lim_{k\to\infty}\sup_{R\in L_{T_k}}\lambda_{T_k}|W(R)|=\lim_{k\to\infty}\sup_{R\in L^*_{T_k}}\lambda_{T_k}|W(R)|=1 \quad \text{a.s.} \tag{1.4.19}$$

最后，我们来填补序列$\{T_k\}$之间的空隙. 我们仅研究"$\sup_{R\in L^*_T}$"情形. 对任给的$T>0$，有$k$使得$T_k<T\leqslant T_{k+1}$. 记$L_T(k)=\{R:R\subset D_T,\ \lambda(R)\leqslant a_T-a_{T_k}\}$.写

$$\begin{aligned}\sup_{R\in L^*_T}\lambda_T|W(R)| \\ \geqslant \sup_{R\in L^*_{T_k}}\lambda_T|W(R)|-4\sup_{R\in L_{T(k)}}\lambda_T|W(R)|.\end{aligned} \tag{1.4.20}$$

由条件 (iii) 易 证$\lambda_{T_k}/\lambda_{T_{k+1}}\to 1(k\to\infty)$.由此并利用条件 (i) 和(1.4.19)式，我们得

$$\lim_{k\to\infty}\sup_{R\in L^*_{T_k}}\lambda_T|W(R)|=1 \qquad \text{a.s.} \tag{1.4.21}$$

现在我们来讨论$\sup_{R\in L_{T(k)}}\lambda_T|W(R)|$.不妨假设$a_T-a_{T_k}>0$. 对$0<\varepsilon<1$，由Csörgö和Révész (1981) 的定理1.12.6有

$$\begin{aligned}P\{\sup_{R\in L_{T(k)}}\lambda_T|W(R)|\geqslant\varepsilon\} \\ \leqslant\frac{c_\varepsilon T}{a_T-a_{T_k}}\left(1+\log\frac{T}{a_T-a_{T_k}}\right)\left(1+\log\frac{b_T}{\sqrt{a_T-a_{T_k}}}\right) \\ \times\exp\left\{-\frac{2\varepsilon^2}{2+\varepsilon}\frac{a_T}{a_T-a_{T_k}}\right. \\ \left.\times\left(\log\frac{T}{a_T}+\log\left(\log\frac{b_T}{\sqrt{a_T}}+1\right)\right)\right\}\end{aligned} \tag{1.4.22}$$

$$
\begin{aligned}
&\leqslant c_{4}\exp\Big\{-\frac{2\varepsilon^{2}}{2+\varepsilon}\frac{a_{T}}{a_{T}-a_{T_{k}}}\log\frac{T}{a_{T}} \\
&\quad+\log\Big(\frac{T}{a_{T}-a_{T_{k}}}\Big(1+\log\frac{T}{a_{T}-a_{T_{k}}}\Big)\Big) \\
&\quad-\frac{2\varepsilon^{2}}{2+\varepsilon}\frac{a_{T}}{a_{T}-a_{T_{k}}}\log\Big(\log\frac{b_{T}}{\sqrt{a_{T}}}+1\Big) \\
&\quad+\log\Big(1+\log\frac{b_{T}}{\sqrt{a_{T}}}+\frac{1}{2}\log\frac{a_{T}}{a_{T}-a_{T_{k}}}\Big)\Big\} \\
&\triangleq c_{4}\exp\{-d_{1}+d_{2}-d_{3}+d_{4}\}.
\end{aligned}
$$

注意到条件（ii）和$T_k$的取法，我们有

$$
\begin{aligned}
(1.4.23)\qquad 1&\geqslant\frac{a_{T_k}}{a_{T_{k+1}}}\geqslant\frac{T_k}{T_{k+1}} \\
&=\Big(1+\frac{1}{\sqrt{k}}\Big)^{k}\Big/\Big(1+\frac{1}{\sqrt{k+1}}\Big)^{k+1} \\
&\geqslant\frac{\sqrt{k+1}}{\sqrt{k+1}+1}\to 1\quad(k\to\infty)
\end{aligned}
$$

和

$$
\begin{aligned}
(1.4.24)\qquad \frac{a_T}{a_T-a_{T_k}}&=\Big(1-\frac{a_{T_k}}{a_T}\Big)^{-1}\geqslant\Big(1-\frac{a_{T_k}}{a_{T_{k+1}}}\Big)^{-1} \\
&\geqslant\Big(1-\frac{\sqrt{k+1}}{\sqrt{k+1}+1}\Big)^{-1}>\sqrt{k}.
\end{aligned}
$$

因此

$$(1.4.25)\qquad d_4=o(d_3)\quad(T\to\infty).$$

又由$\log T(a_T-a_{T_k})^{-1}=\log Ta_T^{-1}+\log a_T(a_T-a_{T_k})^{-1}=o(d_1)$，进一步有

$$(1.4.26)\qquad d_2=o(d_1)\quad(T\to\infty).$$

把（1.4.25）和（1.4.26）式代入（1.4.22）的右边，即知对充分大的$T$，它不超过

$$(1.4.27)\qquad c_4\exp\Big\{-\frac{1}{2}(d_1+d_3)\Big\}$$

$$\leqslant c_\varepsilon \exp\left\{-\frac{\varepsilon^2}{2+\varepsilon}\frac{a_T}{a_T-a_{T_k}}\right.$$

$$\left.\times\left(\log\frac{T}{a_T}+\log\left(\log\frac{b_T}{\sqrt{a_T}}+1\right)\right)\right\}$$

$$\leqslant c_\varepsilon \exp(-\sqrt{k}).$$

由此即得

$$\sum_{k=1}^{\infty} P(\sup_{R\in L_{T}(k)} \lambda_T|W(R)|\geqslant\varepsilon)<\infty.$$

这样就有

$$\text{(1.4.28)}\qquad \varlimsup_{k\to\infty}\sup_{R\in L_{T}(k)}\lambda_T|W(R)|=0 \qquad \text{a.s.}$$

把它和（1.4.21）式代入（1.4.20）式，我们得到

$$\text{(1.4.29)}\qquad \varliminf_{T\to\infty}\sup_{R\in L_T^*}\lambda_T|W(R)|\geqslant 1 \qquad \text{a.s.}$$

注意到，由条件（i）我们有

$$\sup_{R\in L_T}\lambda_T|W(R)|\leqslant \sup_{R\in L_{T_{k+1}}}\lambda_{T_{k+1}}|W(R)|\cdot(\lambda_{T_k}/\lambda_{T_{k+1}}),$$

从条件（iii）和(1.4.19)式即可推出

$$\text{(1.4.30)}\qquad \varliminf_{T\to\infty}\sup_{R\in L_T}\lambda_T|W(R)|\leqslant 1+\varepsilon \qquad \text{a.s.}$$

结合（1.4.29）和（1.4.30）得证定理成立.

作为定理1.4.2的一个推论，我们给出一个关于$W(s,t)$的重对数律，它是（1.4.5）的一个相伴关系.

**推论1.4.2**（Lacey，1989） 我们有

$$\varliminf_{T\to\infty}\sup_{0\leqslant s,t\leqslant T,\,st=T}\frac{|W(s,t)|}{\sqrt{2T\log\log T}}=1 \qquad \text{a.s.}$$

注意，在一参数情形没有这一极限的类似结论。事实上，钟开莱的重对数律（1948）指出

$$\varliminf_{T\to\infty}\left(\frac{8\log\log T}{\pi^2 T}\right)^{1/2}\sup_{0\leqslant t\leqslant T}|W(t)|=1 \qquad \text{a.s.}$$

**推论1.4.3**（孔繁超，1987） 设$a_T$和$\lambda_T$满足定理1.4.2中的(i)，(ii)，(iii) 和定理1.4.1中的 (iv)．那么

$$(1.4.31)\qquad \lim_{T\to\infty}\sup_{R\in L_T}\lambda_T|W(R)|=\lim_{T\to\infty}\sup_{R\in L_T^*}\lambda_T|W(R)|=1\qquad \text{a.s.}$$

**证** 容易看出此时$\delta_T$满足定理1.4.1中所有条件，即$\delta_T$是不增的且当$1<\theta\leqslant\theta_0$时

$$\overline{\lim_{k\to\infty}}\,\delta_{\theta^k}/\delta_{\theta^{k+1}}=\overline{\lim_{k\to\infty}}\,\lambda_{\theta^k}/\lambda_{\theta^{k+1}}\leqslant1+\varepsilon\qquad \text{a.s.}$$

此外，从条件 (iv) 有

$$\lim_{T\to\infty}\lambda_T/\delta_T=1.$$

由此从定理1.4.1和1.4.2即得 (1.4.31) 式．

**注1.4.1** 若定理1.4.1中条件 (iv) 及定理1.4.2中条件 (iv′) 被下述条件 (iv″) 所代替：

$$(\text{iv}'')\qquad \lim_{T\to\infty}(\log Ta_T^{-1}+\log(\log b_Ta_T^{-1/2}+1))/\log\log T=r,\qquad 0\leqslant r<\infty.$$

孔繁超（1987）证明了如下结果：设$0<a_T\leqslant T$，$b_T\geqslant T^{1/2}$是$T$的函数满足定理1.4.1的条件 (i) 和 (ii) 及 (iv″)．那么

$$\begin{cases}\underline{\lim_{T\to\infty}}\sup_{R\in L_T^*}\delta_T|W(R)|=\underline{\lim_{T\to\infty}}\sup_{R\in L_T}\delta_T|W(R)|=\alpha_r & \text{a.s.}\\ \overline{\lim_{T\to\infty}}\sup_{R\in L_T^*}\delta_T|W(R)|=\overline{\lim_{T\to\infty}}\sup_{R\in L_T}\delta_T|W(R)|=1 & \text{a.s.}\\ \underline{\lim_{T\to\infty}}\sup_{R\in L_T^*}\lambda_T|W(R)|=\underline{\lim_{T\to\infty}}\sup_{R\in L_T}\lambda_T|W(R)|=\alpha_r^{-1} & \text{a.s.}\end{cases}$$

其中

$$\alpha_r=\begin{cases}\sqrt{r/(r+1)} & \text{当 } 0\leqslant r<\infty,\\ 1 & \text{当 } r=\infty.\end{cases}$$

### 1.4.2 滞后增量

两参数Wiener过程的滞后增量问题被陆传荣(1990a,b)所讨

论，对应于定理1.1.3的是下述定理.

**定理1.4.3** (Lu, 1990a) 设$b_T \geqslant T^{1/2}$是$T$的不减函数.记

$$L_T^*(t)=\{R: R\subset D_T(b_T),\ \lambda(R)=t\},$$

$$L_T(t)=\{R: R\subset D_T(b_T),\ \lambda(R)\leqslant t\},$$

$$d^*(T,t)=\{2t(\log T/t+\log(\log b_T/t^{1/2}+1)+\log\log t)\}^{1/2}.$$

假设$\gamma_T=d^*(T,T)^{-1}$满足

(ii′) $\gamma_T$是$T$的不增函数；

(iii′) 对任一$\varepsilon>0$存在$\theta_0=\theta_0(\varepsilon)>1$使得对$1<\theta\leqslant\theta_0$有

$$\overline{\lim_{k\to\infty}}\gamma_{\theta^k}/\gamma_{\theta^{k+1}}\leqslant 1+\varepsilon.$$

那么我们有

$$\lim_{T\to\infty}\sup_{0<t\leqslant T}\sup_{R\in L_T^*(t)}|W(R)|/d^*(T,t)=1\quad \text{a.s.}\tag{1.4.32}$$

$$\lim_{T\to\infty}\sup_{0<t\leqslant T}\sup_{R\in L_T(t)}|W(R)|/d^*(T,t)=1\quad \text{a.s.}\tag{1.4.33}$$

**证** 1°为证

$$\overline{\lim_{T\to\infty}}\sup_{0<t\leqslant T}\sup_{R\in L_T(t)}|W(R)|/d^*(T,t)\leqslant 1\quad \text{a.s.}\tag{1.4.34}$$

我们取实数$\theta>1$使$1<2(1+\varepsilon)^2/((2+\varepsilon)\theta)=:1+2\varepsilon'$. 设$1<v\leqslant\theta_0$.定义$T_n=v^n$, $k_n=[((n+1)\log v)/\log\theta]+1$, 又设$t_k$, $k_\theta$ 如定理1.1.3定义.取$\beta=2/\varepsilon'$, $k_n'=[(\log\theta)^{-1}\log(T_{n+1}(\log T_n)^{-\beta})]$.

对任一 $T>0$存在$T_n$使 $T_n<T\leqslant T_{n+1}$.我们有

$$\begin{aligned}&\sup_{0<t\leqslant T}\sup_{R\in L_T(t)}|W(R)|/d^*(T,t)\\ &\leqslant\sup_{-\infty<k\leqslant k_n-1}\sup_{R\in L_{T_{n+1}}(t_k,t_{k+1})}\{2t_k(\log T_n/t_{k+1}\\ &\quad+\log(\log b_{T_n}/t_{k+1}^{1/2}+1)+\log\log t_k)\}^{-1/2}|W(R)|\\ &=:\sup_{-\infty<k\leqslant k_n-1}A_{nk},\end{aligned}\tag{1.4.35}$$

其中

$$L_{T_{n+1}}(t_k,t_{k+1})=\{R: R\subset D_{T_{n+1}}(b_{T_{n+1}}),\ t_k\leqslant\lambda(R)=t\leqslant t_{k+1},\ t\leqslant T_{n+1}\}.$$

注意到

(1.4.36) $$L_{T_{n+1}}(t_k,\ t_{k+1})\subset L_{T_{n+1}}(t_{k+1}),$$

并利用Csörgő和Révész (1981) 的定理 1.12.6 及条件 (iii′)，对充分大的$n$我们有

(1.4.37) $$P\{A_{nk}\geqslant 1+\varepsilon\}$$
$$\leqslant P\Big\{\sup_{R\in L_{T_{n+1}}(t_{k+1})}|W(R)|/t_{k+1}^{1/2}\geqslant(1+\varepsilon)$$
$$\times(2\theta^{-1}(\log T_n/t_{k+1}+\log(\log b_{T_n}/t_{k+1}^{1/2}+1)$$
$$+\log\log t_k))^{1/2}\Big\}$$
$$\leqslant c\frac{T_{n+1}}{t_{k+1}}\Big(1+\log\frac{T_{n+1}}{t_{k+1}}\Big)\Big(1+\log\frac{b_{T_{n+1}}}{\sqrt{t_{k+1}}}\Big)$$
$$\times\exp\Big\{-\frac{2(1+\varepsilon)^2}{(2+\varepsilon)\theta}\Big(\log\frac{Tn}{t_{k+1}}+\log\Big(\log\frac{b_{T_n}}{\sqrt{t_{k+1}}}+1\Big)$$
$$+\log\log t_k\Big)\Big\}$$
$$\leqslant c\Big(\frac{T_{n+1}}{t_{k+1}}\Big)^{-\varepsilon'}\Big(1+\log\frac{T_{n+1}}{t_{k+1}}\Big)(\log T_n)^{\varepsilon}(\log t_k)^{-1-2\varepsilon'}.$$

仿照定理1.1.3的证明，我们有

$$\sum_{n=1}^{\infty}P\Big\{\sup_{-\infty<k\leqslant k_n-1}A_{nk}\geqslant 1+\varepsilon\Big\}<\infty,$$

由Borel-Cantelli引理和 (1.4.35) 即得 (1.4.34) 成立。

2° 为证明 (1.4.32) 和 (1.4.33)，我们只需证

(1.4.38) $$I:=\lim_{T\to\infty}\sup_{0<t\leqslant T}\sup_{R\in L_T^*(t)}|W(R)|/d^*(T,t)\geqslant 1 \quad \text{a.s.}$$

成立。记$n=[T]$，

$$A_{i+1}=\Big[\Big(\frac{n-1}{n}\Big)^{i+1}b_n,\ \Big(\frac{n-1}{n}\Big)^{i}b_n\Big]\times\Big[0,\ \frac{n^{i+1}}{(n-1)^i b_n}\Big]$$
$$i=0,1,\cdots,l,$$

其中$l=\max\{i:n^{i+1}<(n-1)^i b_n^2\}$。显然地，$A_i\in L_n^*(1)$，$1\leqslant i\leqslant l+1$，且$l\sim cn\log b_n^2/n$，我们有

$$I \geqslant \varlimsup_{T \to \infty} \sup_{R \in L_T^*(1)} |W(R)|/d^*(T,1)$$

$$\geqslant \varlimsup_{n \to \infty} \sup_{R \in L_n^*(1)} \{2(\log(n+1)+\log(\log b_{n+1}+1))\}^{-1/2}|W(R)|$$

$$\geqslant \varlimsup_{n \to \infty} \max_{1 \leqslant i \leqslant l+1} \{2(\log(n+1)+\log(\log b_{n+1}+1))\}^{-1/2}|W(A_i)|.$$

由熟知的正态分布尾概率估计(1.1.38)，即得

$$\sum_{n=1}^{\infty} P\{\max_{1 \leqslant i \leqslant l+1} |W(A_i)| \leqslant (2(1-\varepsilon)(\log(n+1)$$

$$+\log(\log b_{n+1}+1)))^{1/2}\}$$

$$\leqslant \sum_{n=1}^{\infty} \{1-\exp(-(1-\varepsilon)(\log(n+1)$$

$$+\log(\log b_{n+1}+1)))\}^{\varepsilon n \log(b_n^2/n)}$$

$$\leqslant \sum_{n=1}^{\infty} \exp\{-c(n\log b_n)^{\varepsilon/2}\} < \infty.$$

因此按Borel–Cantelli引理即可推得（1.4.38）成立，定理1.4.3证毕.

### 1.4.3 增量的一般形式

现在我们希望获得类似于定理1.1.4的一般结果.下述定理不仅蕴含定理1.4.1等，而且很大程度上减弱了所附加的条件.

**定理1.4.4**（Lin，Lu 1990） 设 $a_T$，$d_T$和 $c_T$ 是$T$的非负函数满足$a_T+d_T \geqslant c_T \to \infty$（当$T \to \infty$），且设$b_T(\geqslant T^{1/2})$是$T$的不减函数满足条件（ii′）和（iii′）.若存在常数$A>0$使对任一$T>2$有

(1.4.39) $$a_T+d_T \leqslant A(a_{T-1}+d_{T-1}),$$

那么

(1.4.40) $$\varlimsup_{T \to \infty} \sup_{0 \leqslant t} \sup_{s<t} \sup_{R \in L_{t+s}(s)} |W(R)|/d^*(t+sVc_T, s)=1$$

a.s.

(1.4.41) $\overline{\lim_{T\to\infty}} \sup_{R\in L^*_{a_T+d_T}(a_T)} \beta^*(a_T+d_T,\ a_T)|W(R)|=1$ a.s.

其中

$$\beta^*(M,\ m)=\{2m(\log M/m+\log(\log b_M/m^{1/2}+1) + \log\log M)\}^{-1/2}.$$

进一步，若对任一$0<\varepsilon<1$

(1.4.42)

$$\sum_{n=1}^{\infty} \exp\left\{-\left(\frac{a_n+d_n}{a_n}\left(\log\frac{\bar{b}_n}{\sqrt{a_n}}+1\right)\right)^{\varepsilon} \Big/ (\log(a_n+d_n))^{1-\varepsilon}\right\}<\infty,$$

其中$\bar{b}_n=b_{a_n+d_n}$，且

(1.4.43) $$\lim_{T\to\infty} a_T/a_{[T]}=\lim_{T\to\infty} d_T/d_{[T]}=1,$$

那么

(1.4.44) $$\lim_{T\to\infty} \sup_{s\leq t} \sup_{t\leq T} \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T,\ s)=1$$ a.s.

(1.4.45) $$\lim_{T\to\infty} \sup_{0\leq t\leq d_T} \sup_{R\in L^*_{t+a_T}(a_T)} \beta^*(a_T+d_T,\ a_T)|W(R)|=1$$ a.s.

**注1.4.3** 在(1.4.41)中取$d_T=T-a_T$，我们有

(1.4.46) $$\overline{\lim_{T\to\infty}} \sup_{R\in L^*_T} \delta_T|W(R)| = \overline{\lim_{T\to\infty}} \sup_{R\in L^*_T(a_T)} \beta^*(T,a_T)|W(R)|=1 \quad \text{a.s.}$$

此时在$a_T$上不附设任何条件.结合(1.4.40)可推得在条件(ii′)和(iii′)下(1.4.3)成立.即对于定理1.4.1来说$a_T$是不减的这一条件并非必要，可以去掉，且条件(ii)和(iii)可被较弱的条件(ii′)和(iii′)所代替.若附设对任一$0<\varepsilon<1$我们有

(1.4.42′) $$\sum_{n=1}^{\infty} \exp\left\{-\left(\frac{n}{a_n}\left(\log\frac{b_n}{\sqrt{a_n}}+1\right)\right)^{\varepsilon} \Big/ (\log n)^{1-\varepsilon}\right\}<\infty$$

且

(1.4.43′) $$\lim_{T\to\infty} a_T/a_{[T]}=1,$$

那么（1.4.44）也正确。

显然（1.4.40）蕴含

(1.4.47) $$\overline{\lim_{T\to\infty}}\ \sup_{0\leqslant t\leqslant T-a_T}\ \sup_{R\in L_{t+a_T}(a_T)}|W(R)|/d^*(t+a_T,a_T)\leqslant 1 \quad \text{a.s.}$$

(1.4.48) $$\overline{\lim_{T\to\infty}}\ \sup_{0\leqslant t\leqslant T-a_T}\ \sup_{R\in L^*_{t+a_T}(a_T)}|W(R)|/d^*(t+a_T,a_T)\leqslant 1 \quad \text{a.s.}$$

又若条件（ii′），（iii′）和条件（a）$a_T\to\infty(T\to\infty)$ 被满足，我们有

(1.4.49) $$\overline{\lim_{T\to\infty}}\ \sup_{0\leqslant t\leqslant T-a_T}\ \sup_{R\in L_{t+a_T}(a_T)}|W(R)|/d^*(t+a_T,a_T)=1 \quad \text{a.s}$$

(1.4.50) $$\overline{\lim_{T\to\infty}}\ \sup_{0\leqslant t\leqslant T-a_T}\ \sup_{R\in L^*_{t+a_T}(a_T)}|W(R)|/d^*(t+a_T,a_T)=1 \quad \text{a.s.}$$

进一步，若附设条件(1.4.42′)和(1.4.43′)也被满足，则(1.4.49)和(1.4.50)中$\overline{\lim}$ 可被lim代替。这些结果改进了陆传荣（1990a）的一个定理，它们都是Hanson和Russo(1983a) 的定理3.2B在两参数情形的一个类比。

在定理1.4.4的证明中需要下述引理。

**引理1.4.1** 假设$\gamma_T$满足定理1.4.3的条件（ii′）和（iii′）。那么我们有

(1.4.51) $$\lim_{a\to\infty}\ \sup_{\substack{R\in L_v(v-u)\\ a\leqslant v-u}}|W(R)|/d^*(v,v-u)=1 \quad \text{a.s.}$$

引理的证明十分类似于推论1.1.1，故不再陈述于此。

定理1.4.4的证明。

1° 我们来证

$$(1.4.52)\quad \overline{\lim_{T\to\infty}}\ \sup_{0\le t}\ \sup_{0<s}\ \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T,s)\le 1 \quad \text{a.s.}$$

显然地，我们可设$c_T$是不减的，若不然，我们可用

$$c_T^*=\inf_{T\le t} c_t \quad 代替 c_T.$$

对任给的$B>0$，我们可写

$$\begin{aligned}
&\sup_{0\le t}\ \sup_{0<s}\ \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T,s)\\
&=(\sup_{0\le t}\ \sup_{B<s}\ \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T,s))\\
&\quad\vee(\sup_{0\le t\le 1}\ \sup_{0<s\le B}\ \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T,s))\\
&\quad\vee(\sup_{1\le t}\ \sup_{0<s\le B}\ \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T,s))\\
&=:I_1\vee I_2\vee I_3.
\end{aligned}$$

应用引理1.4.1，对任给的$\varepsilon>0$存在充分大的$B=B(\varepsilon)$，使得$I_1\le 1+\varepsilon$ a.s. 设$\theta>1$（下面取定）.应用Csörgő和Révész（1981）的定理1.12.6和条件（iii′），对充分大的$T$我们有

$$\begin{aligned}
&(1.4.53)\ P\{I_3\ge 1+\varepsilon\}\\
&\le\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P\{\sup_{\theta^i\le t\le\theta^{i+1}}\ \sup_{B\theta^{-(j+1)}<s\le B\theta^{-j}}\ \sup_{R\in L_{t+s}(s)}\\
&\quad |W(R)|/d^*(t+s\vee c_T,s)\ge 1+\varepsilon\}\\
&\le\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P\Big\{\sup_{R\in L_{\theta^{i+1}+B\theta^{-j}}(B\theta^{-j})}|W(R)|\ge(1+\varepsilon)\\
&\quad\Big\{2B\theta^{-(j+1)}\Big(\log\frac{\theta^i+c_T}{B\theta^{-j}}+\log\Big(\log\frac{\theta_\theta{}^{i}+B\theta^{-(j+1)}}{\sqrt{B\theta^{-j}}}+1\Big)\\
&\quad+\log\log B\theta^{-(j+1)}\Big)\Big\}^{1/2}\Big\}\\
&\le c\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Big\{\frac{\theta^{i+1}+B\theta^{-j}}{B\theta^{-j}}\Big(1+\log\frac{\theta^{i+1}+B\theta^{-j}}{B\theta^{-j}}\Big)
\end{aligned}$$

$$\left(1+\log\frac{b_{\theta^{j+1}+B\theta^{-j}}}{\sqrt{B\theta^{-j}}}\right)\left\{\frac{\theta^{i}+c_T}{B\theta^{-j}}\left(1+\log\frac{b_{\theta^{i}+B\theta^{-(j+1)}}}{\sqrt{B\theta^{-j}}}\right)\right.$$

$$\left.\times\log B\theta^{-(j+1)}\Big\}^{-(1+\varepsilon)}\right\}$$

$$\leqslant c\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(\theta^{i}+c_T)^{-\varepsilon/2}\theta^{-j\varepsilon/2}\leqslant cc_T^{-\varepsilon/2}\to 0\quad(T\to\infty),$$

类似地，我们有$P\{I_2\geqslant 1+\varepsilon\}\to 0(T\to\infty)$。由于$I_2$和$I_3$都是$T$的不增函数，故得

$$\overline{\lim_{T\to\infty}}\ (I_2\vee I_3)\leqslant 1+\varepsilon\qquad \text{a.s.}$$

这就证明了（1.4.52）式。

2° 现在为证（1.4.40）和（1.4.41），我们只需证明

$$(4.54)\qquad \overline{\lim_{T\to\infty}}\sup_{R\in L^{*}_{a_T+d_T}(a_T)}\beta^{*}(a_T+d_T,a_T)|W(R)|\geqslant 1\quad \text{a.s.}$$

定义$N_1=1$,

$$N_k=\max\{n:\ a_n+d_n\leqslant\theta^k\}\qquad k\geqslant 2.$$

由条件（1.4.39）我们有

$$(1.4.55)\qquad \theta^k/A<(a_{N_k+1}+d_{N_k+1})/A\leqslant a_{N_k}+d_{N_k}\leqslant\theta^k.$$

首先，我们假设

$$\rho_1:=\overline{\lim_{N\to\infty}}\ a_N/(a_N+d_N)<1.$$

令$L=L(k)$是使下式成立的最大整数：

$$(a_{N_k}+d_{N_k})^{L+1}/(d_{N_k}^{L}\tilde{b}_{N_k})\leqslant\tilde{b}_{N_k}\qquad k=2,3,\ \cdots$$

又令矩形

$$S_i(k)=[x_1(i),x_2(i)]\times[y_1(i),y_2(i)]$$

$$=\left[\left(\frac{d_{N_k}}{a_{N_k}+d_{N_k}}\right)^{i+1}\tilde{b}_{N_k},\left(\frac{d_{N_k}}{a_{N_k}+d_{N_k}}\right)^{i}\tilde{b}_{N_k}\right]$$

$$\times\left[\frac{(a_{N_{k-1}}+d_{N_{k-1}})(a_{N_k}+d_{N_k})^{i+1}}{d_{N_k}^{i+1}\tilde{b}_{N_k}},\right.$$

$$\left.\frac{(a_{N_k}+d_{N_k})^{i+1}}{d_{N_k}^{i}\tilde{b}_{N_k}}\right] \qquad i=0,1,\cdots,\ L.$$

由（1.4.55）及假设$\rho_1<1$，对任给$\varepsilon>0$存在$k_0>1$使对某$0<\delta<1-\rho_1$和$k\geqslant k_0$，

$$(a_{N_{k-1}}+d_{N_{k-1}})/d_{N_k}\leqslant\theta^{k-1}/(\theta^k(1-\rho_1-\delta)/A)<\varepsilon.$$

注意到

$$a_{N_{k-1}}+d_{N_{k-1}}=x_1(i)y_1(i)<x_2(i)y_2(i)=a_{N_k}+d_{N_k},$$

且当$\theta$充分大时

$$0<x_1(i)<x_2(i)\leqslant\tilde{b}_{N_k},\quad 0<y_1(i)<y_2(i)\leqslant\tilde{b}_{N_k},$$

$i=0,1,\cdots,L$.记$\bar{D}_{N_k}=D_{a_{N_k}+d_{N_k}}$.这样我们有

$$S_i(k)\subset\bar{D}_{N_k}-\bar{D}_{N_{k-1}},$$

且对每一$k$，$S_i(k)$，$i=0,1,\cdots,L$是互不相交的矩形.所以对给定的$0<\varepsilon<1$，对$k\geqslant k_0$和每一$i$，当$\theta$充分大时我们有

$$(1.4.56)\qquad (1-\varepsilon)a_{N_k}\leqslant\lambda(S_i(k))\leqslant a_{N_k}.$$

因此，

$$(1.4.57)\quad P\{\max_{0\leqslant i\leqslant L}\beta^*(a_{N_k}+d_{N_k},\ a_{N_k})|W(S_i(k))|\geqslant 1-\varepsilon\}$$

$$\geqslant 1-\Bigg(1-2\Bigg\{1-\Phi\Bigg(\sqrt{1-\varepsilon}\Bigg(2\Bigg(\log\frac{a_{N_k}+d_{N_k}}{a_{N_k}}+$$

$$\log\Bigg(\log\frac{\tilde{b}_{N_k}}{\sqrt{a_{N_k}}}+1\Bigg)+\log\log(a_{N_k}+d_{N_k})\Bigg)\Bigg)^{1/2}\Bigg)\Bigg\}\Bigg)^{L+1}$$

$$\geqslant 1-\Bigg\{1-\Bigg(\frac{a_{N_k}+d_{N_k}}{a_{N_k}}\Bigg(\log\frac{\tilde{b}_{N_k}}{\sqrt{a_{N_k}}}+1\Bigg)$$

$$\times\log(a_{N_k}+d_{N_k})\Bigg)^{-(1-\varepsilon)}\Bigg\}^{L+1}$$

$$\geqslant 1-\exp\Bigg\{-\Bigg(\frac{a_{N_k}}{a_{N_k}+d_{N_k}}\cdot\frac{1}{\log\tilde{b}_{N_k}/\sqrt{a_{N_k}}+1}$$

$$\times\frac{1}{\log(a_{N_k}+d_{N_k})}\Bigg)^{1-\varepsilon}(L+1)\Bigg\}.$$

由$L$的定义可知 $L+1\geqslant c\dfrac{a_{N_k}+d_{N_k}}{a_{N_k}}\log\dfrac{\tilde{b}^2_{N_k}}{a_{N_k}+d_{N_k}}$，于是我们有

$$(1.4.58)\quad \sum_{k=1}^{\infty} P\{\max_{0\leqslant i\leqslant L}\beta^*(a_{N_k}+d_{N_k},\ a_{N_k})|W(S_i(k))|\geqslant 1-\varepsilon\}$$

$$\geqslant c\sum_{k=1}^{\infty}\left(\frac{a_{N_k}+d_{N_k}}{a_{N_k}}\right)^{\varepsilon}\frac{1}{\log(a_{N_k}+d_{N_k})}$$

$$\times\left\{\log\frac{\tilde{b}^2_{N_k}}{a_{N_k}+d_{N_k}}\Big/\left(\log\frac{\tilde{b}^2_{N_k}}{a_{N_k}+d_{N_k}}+\log\frac{a_{N_k}+d_{N_k}}{a_{N_k}}\right)\right\}$$

$$\geqslant c\sum_{k=1}^{\infty}\frac{1}{\log(a_{N_k}+d_{N_k})}=\infty,$$

由此及对每一$k$，$W(S_i(k))$的独立性和Borel-Cantelli引理即可推得

$$\overline{\lim_{k\to\infty}}\max_{0\leqslant i\leqslant L}\beta^*(a_{N_k}+d_{N_k},\ a_{N_k})|W(S_i(k))|\geqslant 1-\varepsilon\quad \text{a.s.}$$

记$a'_{N_k}=\lambda(S_i(k))$．由(1.4.56)知$\bar{a}_{N_k}:=a_{N_k}-a'_{N_k}\leqslant\varepsilon a_{N_k}$．那么

$$\overline{\lim_{T\to\infty}}\sup_{R\in L^*_{a_T+d_T}(a_T)}\beta^*(a_T+d_T,\ a_T)|W(R)|$$

$$\geqslant\overline{\lim_{k\to\infty}}\left\{\max_{0\leqslant i\leqslant L}\beta^*(a_{N_k}+d_{N_k},\ a_{N_k})|W(S_i(k))|\right.$$
$$\left.-2\sup_{R\in L_{a_{N_k}+d_{N_k}}(\tilde{a}_{N_k})}\beta^*(a_{N_k}+d_{N_k},\ a_{N_k})|W(R)|\right\}$$

$$\geqslant 1-3\varepsilon\qquad \text{a.s.}$$

其中我们应用了类似于（1.4.52）的结论．这样当$\rho_1<1$时得证(1.4.54)成立．

其次，考察$\rho_2:=\lim\limits_{N\to\infty}a_N/(a_N+d_N)=1$情形．定义$L=L(k)$是使下式成立的最大整数：

$$(a_{N_k}+d_{N_k})^{1/2}M^{L+1}\leqslant\tilde{b}_{N_k},$$

其中$M(=2/\varepsilon)>1$，又定义矩形

$$S_i(k)=[x_1(i),\ x_2(i)]\times[y_1(i),\ y_2(i)]$$

$$=[(a_{N_k}+d_{N_k})^{1/2}M^i,\ (a_{N_k}+d_{N_k})^{1/2}M^{i+1}]$$
$$\times[(a_{N_{k-1}}+d_{N_{k-1}})(a_{N_k}+d_{N_k})^{-1/2}M^{-i},$$
$$(a_{N_k}+d_{N_k})^{1/2}M^{-i-1}],\qquad i=0,1,\cdots,L.$$

注意到此时

$$a_{N_{k-1}}+d_{N_{k-1}}=x_1(i)y_1(i)<x_2(i)y_2(i)=a_{N_k}+d_{N_k},$$
$$0<x_1(i)<x_2(i)\leqslant\tilde{b}_{N_k},\quad 0<y_1(i)<y_2(i)\leqslant\tilde{b}_{N_k},$$

$i=0,1,\cdots,L$. 因此$S_i(k)\subset\bar{D}_{N_k}-\bar{D}_{N_{k-1}}$，且对每一$k$，$S_i(k)$ 是互不相交的矩形。由假设$\rho_2=1$，仿照（1.4.56）的证明，假如$\theta$充分大，那么对充分大的$k$和每一$i$我们有

$$(1-\varepsilon)a_{N_k}\leqslant(1-\varepsilon)(a_{N_k}+d_{N_k})\leqslant\lambda(S_i(k))$$
$$\leqslant\left(1-\frac{\varepsilon}{2}\right)(a_{N_k}+d_{N_k})$$
$$\leqslant a_{N_k}.$$

由此仿照上面$\rho_1<1$情形继续讨论之就可给出此时（1.4.54）仍成立。

第三，若$\rho_1=1$且$\varliminf\limits_{N\to\infty}a_N/(a_N+d_N)<1$，那么存在一个正整数列$G=\{N''\}$使得

$$\lim_{N''\to\infty}a_{N''}/(a_{N''}+d_{N''})=1\quad 且\quad \varlimsup_{\substack{N'\to\infty\\ N'\notin G}}a_{N'}/(a_{N'}+d_{N'})<1.$$

结合上面对于$\rho_1<1$和$\rho_2=1$的讨论，下述两个不等式至少有一成立：

$$\sum_{N_k\in G}P\{\max_{0\leqslant i\leqslant L}\beta^*(a_{N_k}+d_{N_k},\ a_{N_k})|W(S_i(k))|\geqslant1-\varepsilon\}=\infty$$

和

$$\sum_{N_k\notin G}P\{\max_{0\leqslant i\leqslant L}\beta^*(a_{N_k}+d_{N_k},\ a_{N_k})|W(S_i(k))|\geqslant1-\varepsilon\}=\infty.$$

因此，这时（1.4.54）式也成立。由此推得（1.4.40）和（1.4.41）式成立。

3° 假设条件（1.4.42）和（1.4.43）被满足。我们来证

$$(1.4.59)\ \varlimsup_{T\to\infty}\ \sup_{t\leqslant t\leqslant d_T}\ \sup_{R\in L^*_{t+a_T}(a_T)}\ \beta^*(a_T+d_T,\ a_T)|W(R)|\geqslant 1\quad \text{a.s.}$$

它与（1.4.40）相结合即得证（1.4.44）和（1.4.45）成立。

设$L=L(N)$是使下式成立的最大整数：

$$(a_N+d_N)^{L+1}/d_N^L\ \tilde{b}_N\leqslant\tilde{b}_N\qquad 当\rho_1<1,$$

$$(a_N+d_N)^{1/2}M^{L+1}\leqslant\tilde{b}_N\qquad 当\rho_2=1.$$

并定义矩形

$$S_i(N)=\begin{cases}\left[\left(\dfrac{d_N}{a_N+d_N}\right)^{i+1}\tilde{b}_N,\left(\dfrac{d_N}{a_N+d_N}\right)^{i}\tilde{b}_N\right]\\ \times\left[0,\dfrac{(a_N+d_N)^{i+1}}{d_N^i\ \tilde{b}_N}\right] & 当\rho_1<1,\\ [(a_N+d_N)^{1/2}M^i,\ (a_N+d_N)^{1/2}M^{i+1}]\\ \times[0,(a_N+d_N)^{1/2}M^{-i-1}] & 当\rho_2=1,\end{cases}$$

其中$i=0,\ 1,\ \cdots,\ L$.我们注意到

$$\lambda(S_i(N))=\begin{cases}a_N & 当\rho_1<1,\\ (1-\varepsilon)(a_N+d_N) & 当\rho_2=1;\end{cases}$$

$$L\geqslant\begin{cases}c\dfrac{a_N+d_N}{a_N}\log\dfrac{\tilde{b}_N^2}{a_N+d_N} & 当\rho_1<1,\\ c\log\dfrac{\tilde{b}_N^2}{a_N+d_N} & 当\rho_2=1.\end{cases}$$

由于集$S_i(N)$，$i=0,\ 1,\ \cdots,\ L$，是互不相交的，所以当$N$充分大时我们有

$$P\{\max_{0\leqslant i\leqslant L}\beta^*(a_N+d_N,\ a_N)|W(S_i(N))|\leqslant 1-\varepsilon\}$$

$$\leqslant\left\{1-\exp\left(-(1-\varepsilon)\left(\log\frac{a_N+d_N}{a_N}+\log\left(\log\frac{\tilde{b}_N}{\sqrt{a_N}}+1\right)\right.\right.\right.$$

$$\left.\left.\left.+\log\log(a_N+d_N)\right)\right)\right\}^{L+1}$$

$$\leqslant\exp\left\{-\left(\frac{a_N+d_N}{a_N}\left(\log\frac{\tilde{b}_N}{\sqrt{a_N}}+1\right)\right)^{\varepsilon/2}(\log(a_N+d_N))^{-1+\varepsilon/2}\right\}.$$

利用条件（1.4.42）并仿照从（1.4.57）获得(1.4.54)的证明，我们得

$$(1.4.60)\ \varliminf_{N\to\infty}\ \max_{0\leqslant n\leqslant d_N}\ \sup_{R\in L^*_{n+a_N}(a_N)} \beta^*(a_N+d_N,a_N)|W(R)|\geqslant 1-\varepsilon$$

我们可写　　a.s.

$$(1.4.61)\ \sup_{0\leqslant t\leqslant d_T}\ \sup_{R\in L^*_{t+a_T}(a_T)} \beta^*(a_T+d_T,\ a_T)|W(R)|$$

$$\geqslant \max_{0\leqslant n\leqslant d_N}\ \sup_{R\in L^*_{n+a_{(T)}}(a_{(T)})} \beta^*(a_T+d_T,\ a_T)|W(R)|$$

$$-\sup_{0\leqslant t\leqslant d_T}\ \sup_{R\in L_{t+a_t}(a_T-a_{(T)})} \beta^*(a_T+d_T,\ a_T)|W(R)|$$

$$=: J_1-J_2.$$

回顾（1.4.52）的证明并利用条件（1.4.43），我们有 $\varlimsup_{T\to\infty} J_2=0$ a.s.，综合条件(1.4.43)和（iii′）我们可证明

$$\beta^*(a_T+d_T,a_T)/\beta^*(a_{(T)}+d_{(T)},\ a_{(T)})\to 1\quad(T\to\infty).$$

因此从(1.4.60)可得 $\varliminf_{T\to\infty} J_1\geqslant 1-\varepsilon$ a.s.，这就得证(1.4.59)成立.定理证毕.

## § 1.5　非平稳Gauss过程的增量

设$\{X(t);\ t\geqslant 0\}$是具有平稳增量的Gauss过程，$EX(t)=0$，$X(0)=0$ a.s.，假设

$$(1.5.1)\ \sigma^2(h)=E(X(t+h)-X(t))^2=EX^2(h)=C_0h^{2\alpha},$$

其中常数$C_0>0$，$0<\alpha<1$.这一条件蕴含着过程的样本轨道概率为1地是连续的（见Fernique 1964）.

我们自然会问：陈述于§1.1和§1.2中的关于Wiener过程增量的一些结果对于Gauss过程$\{X(t)\}$是否仍成立.回答是：很多结果是成立的.

### 1.5.1 Gauss过程$\{X(t)\}$的增量

定义

$$H(T,h)=\sup_{0\leqslant t\leqslant T-h}\ \sup_{0\leqslant s\leqslant h}|X(t+s)-X(t)|,$$

$$I(T,h)=\sup_{0\leqslant t<t'\leqslant T}\ \sup_{t'-t\leqslant h}|X(t')-X(t)|.$$

Ortega(1984) 证明着对应于定理1.1.1的下述结果。

**定理1.5.1** (Ortega, 1984) 设$0<a_T\leqslant T$是$T$的函数，满足定理1.1.1的条件 (i) 和 (ii)。那么

$$\text{(1.5.2)}\qquad \overline{\lim_{T\to\infty}}\beta_T I(T,a_T)=\overline{\lim_{T\to\infty}}\beta_T H(T,a_T)$$

$$=\overline{\lim_{T\to\infty}}\beta_T|X(T+a_T)-X(T)|=1\quad \text{a.s.}$$

其中

$$\beta_T=\{2\sigma^2(a_T)(\log Ta_T^{-1}+\log\log T)\}^{-1/2}.$$

此外，若还满足定理1.1.1的条件 (iii)，那么

$$\text{(1.5.3)}\qquad \lim_{T\to\infty}\beta_T I(T,a_T)=\lim_{T\to\infty}\beta_T H(T,a_T)$$

$$=\lim_{T\to\infty}\sup_{0\leqslant t\leqslant T-a_T}\beta_T|X(t+a_T)-X(t)|=1\quad \text{a.s.}$$

在定理1.5.1的证明中将用到下述引理。

**引理1.5.1** (Fernique, 1964) 设$\{Y(t);\ t\in[0,1]\}$是可分实Gauss过程满足

$$\mathrm{E}(Y(t)-Y(s))^2\leqslant A^2(|t-s|),$$

其中$A(x)$是连续、不减的，且满足$\int_1^\infty A(e^{-u^2})du<\infty$。此外$EY^2(t)\leqslant\Gamma^2(\Gamma>0)$。那么对$x\geqslant(4\log a)^{1/2}$我们有

$$P\left\{\sup_{0\leqslant t\leqslant 1}|Y(t)|>x\left(\Gamma+4\int_1^\infty A(a^{-u^2})\,du\right)\right\}\leqslant ca^2\int_x^\infty e^{-u^2/2}du,$$

其中$c$是绝对常数且$a\geqslant 2$。

**引理1.5.2** (Ortega, 1984) 设$\{X(t)\}$如定理1.5.1，那么当

$0<h\leqslant T$且$z\geqslant 4$时我们有

$$P\{I(T,h)>z\sigma(h)\}\leqslant c\frac{T}{h}\ \frac{z^{5/\alpha-1}}{(\log z)^{2/\alpha}}e^{-z^2/2}.$$

**证** 给定$\delta>0$，$x\geqslant e$且定义$N=(2/\delta)^{1/\alpha}/h$.那么

(1.5.4) $P\{I(T,h)\geqslant(1+\delta)x\sigma(h)\}$

$$\leqslant P\left\{\max_{0\leqslant i\leqslant [NT]}\sup_{0\leqslant s\leqslant h}\left|X\left(\frac{i}{N}+s\right)-X\left(\frac{i}{N}\right)\right|>x\sigma(h)\right\}$$

$$+P\left\{\max_{0\leqslant i\leqslant [NT]}\sup_{0\leqslant s\leqslant 1/N}\left|X\left(\frac{i}{N}+s\right)-X\left(\frac{i}{N}\right)\right|\right.$$

$$\left.>x\sigma(h)\delta/2\right\}$$

$$\leqslant(NT+1)(P\{\sup_{\leqslant s\leqslant h}|X(s)|>x\sigma(h)\}$$

$$+P\{\sup_{0\leqslant s\leqslant 1/N}|X(s)|>x\sigma(1/N)\}).$$

我们将运用引理1.5.1的Fernique不等式来获得上两概率的界.设$\Delta>0$，定义$Y(t)=X(t\Delta)$，$0\leqslant t\leqslant 1$.那么$Y(t)$是一个均值为0的连续 Gauss 过程，满足$E(Y(t)-Y(s))^2\leqslant\sigma^2$ $(|t-s|\Delta)$和$EY^2(t)\leqslant\sigma^2(\Delta)$.所以当$v\geqslant(4\log a)^{1/2}$时我们有

$$P\{\sup_{0\leqslant t\leqslant 1}|Y(t)|\geqslant v\sigma(\Delta)F(a,\Delta)\}\leqslant ca^2\int_v^\infty e^{-u^2/2}du,$$

其中

$$F(a,\Delta)=1+\frac{4}{\sigma(\Delta)}\int_1^\infty\sigma(\Delta a^{-u^2})du\leqslant 1+\frac{2}{\alpha a^\alpha\log a}.$$

所以若$x=v(1+2/\alpha a^\alpha\log a)$，我们就得

$$P\{\sup_{0\leqslant s\leqslant\Delta}|X(s)|>x\sigma(\Delta)\}\leqslant ca^2\int_v^\infty e^{-u^2/2}du$$

$$\leqslant ca^2\psi(x)\exp\left(\frac{2\alpha x^2a^\alpha\log a+2x^2}{(2+\alpha a^\alpha\log a)^2}\right),$$

其中$\psi(x)=\frac{1}{x\sqrt{2\pi}}e^{-x^2/2}$。现在记$a=x^{2/\alpha}/(\log x)^{1/\alpha}$，那么$v\geqslant(4\log a)^{1/2}$且$2x^2/(2+aa^\alpha\log a)^2\to 0$。指数中余下的项趋于1（当$x\to\infty$时）所以

$$P\{\sup_{0\leqslant s\leqslant\Delta}|X(s)|>x\sigma(\Delta)\}\leqslant c\frac{x^{4/\alpha}}{(\log x)^{2/\alpha}}\psi(x).$$

在（1.5.4）式右边运用这一不等式两次我们得

$$P\{I(T,h)>(1+\delta)x\sigma(h)\}$$
$$\leqslant cNT\ \frac{x^{4/\alpha}}{(\log x)^{2/\alpha}}\psi(x)=c\frac{T}{h\delta^{1/\alpha}}\ \frac{x^{4/\alpha-1}}{(\log x)^{2/\alpha}}e^{-x^2/2}.$$

现在令$z=x(1+\delta)$，那么

$$P\{I(T,h)>z\sigma(h)\}$$
$$\leqslant c\frac{T}{h}\ \frac{z^{4/\alpha-1}}{\delta^{1/\alpha}(\log z)^{2/\alpha}}e^{-z^2/2}\cdot\exp\left(\frac{2\delta z+z\delta^2}{2(1+\delta)^2}\right).$$

最后，选$\delta=1/z$，得证引理1.5.2成立。

**引理1.5.3**(Berman，1964； Slepian，1962)设$\{X_j;j=1,2,\cdots,n\}$是中心化的平稳Gauss随机变量序列，$EX_j^2=1,EX_iX_j=r_{ij}$。记$I_c^{+1}=[c,\infty),I_c^{-1}=(-\infty,c)$。若$c_j\in R(j=1,2,\cdots,n)$，用$F_j$记事件$\{X_j\in I_{c_j}^{\varepsilon_j}\}$，其中$\varepsilon_j$或为$+1$或为$-1$。设$K\subset\{1,\cdots,n\}$，那么：

(i) 当$\varepsilon_i\varepsilon_j=1$时，$P\left\{\bigcap_{j\in K}F_j\right\}$是$r_{ij}$的增函数，当$\varepsilon_i\varepsilon_j=-1$时，则是$r_{ij}$的减函数；

(ii) 若$\{K_l,l=1,2,\cdots,s\}$是$K$的一个分划，那么

$$\left|P\left\{\bigcap_{j\in K}F_j\right\}-\prod_{l=1}^{s}P\left\{\bigcap_{j\in K_l}F_j\right\}\right|$$
$$\leqslant\sum_{1\leqslant l<m\leqslant s}\sum_{j\in K_l}\sum_{i\in K_m}|r_{ij}|\varphi(c_i,c_j;r_{ij}^*),$$

其中$\varphi(x,y;r)$是具有相关系数$r$的标准二元正态密度，数$r_{ij}^*$是0和$r_{ij}$之间的一个数。

引理1.5.4　设$\{A_n; n\geq 1\}$是事件序列，若

(i) $\sum_{n=1}^{\infty} P(A_n)=\infty$;

(ii) $$\lim_{n\to\infty} \sum_{1\leq j<k\leq n} (P(A_jA_k)-P(A_j)P(A_k)) \Big/ \left(\sum_{j=1}^{n} P(A_j)\right)^2=0,$$

那么$P(A_n \text{ i.o.})=1$.

这些引理的证明此处从略.引理1.5.4的证明可在Billingsley (1986) 中找到.

定理1.5.1的证明

1°利用引理1.5.2并仿照定理1.1.1的证明（见Csörgö和Révész(1981)或陆传荣等(1989)第五章定理2.1的证明），我们也可证得

$$\overline{\lim_{T\to\infty}}\beta_T I(T, a_T)\leq 1+\varepsilon \quad \text{a.s.} \tag{1.5.5}$$

2°其次往证

$$\overline{\lim_{T\to\infty}}\beta_T |X(T)-X(T-a_T)|\geq 1 \quad \text{a.s.} \tag{1.5.6}$$

记$\rho=\lim_{T\to\infty} a_T/T$.若$\rho=1$，那么必有$a_T=T$且$|X(T)-X(T-a_T)|=|X(T)|$.故由重对数律（Orey1971）得（1.5.6）式成立.若$\rho<1$，定义$T_1=1$，$T_k-a_{T_k}=T_{k-1}$ $(k>1)$,记

$$Y_k:=(X(T_k)-X(T_k-a_{T_k}))/\sigma(a_{T_k}),$$

此时$EY_k=0$，$EY_k^2=1$.对任给的$\varepsilon>0$，设

$$A_n=\{\beta_{T_n}Y_n>(1-\varepsilon)/\sigma(a_{T_n})\}.$$

仿照定理1.1.1的证明.我们有$\sum_{n=1}^{\infty} P(A_n)=\infty$. 所以为证(1.5.6)成立，我们只需验证引理1.5.4的（ii）被满足.

由引理1.5.3，若$EY_jY_k\leq 0$，那么$P(A_jA_k)\leq P(A_j)P(A_k)$.因此(ii)显然成立，这就是说当(1.5.1)中$0<\alpha\leq 1/2$时是正确的.所以余下仅需考察(1.5.1)中$1/2<\alpha<1$情形.设$k\geq j+3$，我

们有

$$EY_jY_k=(P^{2a}+G(Q,R))/(2Q^aR^a),$$

其中

$$P=\sum_{i=j+1}^{k-1}a_{T_i},\quad Q=a_{T_j},\quad R=a_{T_k},$$

$$G(U,V)=(P+U+V)^{2a}-(P+U)^{2a}-(P+V)^{2a}.$$

由Taylor定理

$$G(Q,R)=-P^{2a}+2a(2a-1)P^{2a-2}QR+S,$$

其中对某一$0<\tau<1$

$$S=\frac{1}{3!}2a(2a-1)(2a-2)((Q+R)^3(P+\tau Q+\tau R)^{2a-3}$$

$$-Q^3(P+\tau Q)^{2a-3}-R^3(P+\tau R)^{2a-3}).$$

容易看出

$$S\leqslant\frac{2a(2a-1)(2-2a)}{3!}Q^3(P+\tau Q)^{2a-3},$$

$$G(Q,R)\leqslant-P^{2a}+3a(2a-1)QRP^{2a-2}.$$

由此得到

$$(1.5.7)\quad EY_jY_k\leqslant c(a_{T_j}a_{T_k})^{1-a}\left(\sum_{i=j+1}^{k-1}a_{T_i}\right)^{2(a-1)}.$$

因为$\rho<1$，不失一般性我们可设$a_1<1$，且由$a_1\geqslant a_{T_k}/T_k$，所以$T_k(1-a_1)\leqslant T_{k-1}$，由此可得$a_{T_r}\leqslant(1-a_1)^{-1}a_{T_{k-1}}$且当$k\geqslant j+3$时有

$$EY_jY_k\leqslant c\left(a_{T_j}\Big/\sum_{i=j+1}^{k-1}a_{T_i}\right)^{1-a}\leqslant c(k-j-1)^{a-1}=:\eta_{jk}.$$

现在我们来验证引理1.5.4的（ii）.引理1.5.3指出：

$$P(A_jA_k)-P(A_j)P(A_k)\leqslant r_{jk}\phi(\lambda_j,\lambda_k;\ r^*_{jk}),$$

其中$r_{jk}=EY_jY_k$，$\lambda_k=(1-\varepsilon)\beta_{T_k}^{-1}/\sigma(a_{T_k})$.对 于 固定的某$m$，我们来考察

$$I=\sum_{k=m}^{n}\sum_{j=1}^{k-s}(P(A_jA_k)-P(A_j)P(A_k))$$

$$\leqslant\left(\sum_{k=m}^{n}\sum_{j=1}^{k-\gamma_k}+\sum_{k=m}^{n}\sum_{j=k-\gamma_k}^{k-s}\right)r_{jk}\phi(\lambda_j,\lambda_k;r_{jk}^{*})$$

$$=:I_1+I_2,$$

其中$\gamma_k=[\lambda_k^{4/(1-\alpha)}]$.对于第一项

$$I_1\leqslant\sum_{k=m}^{n}\sum_{j=1}^{k-\gamma_k}\frac{r_{jk}}{2\pi(1-r_{jk}^{*2})^{1/2}}\exp\left\{-\frac{\lambda_j^2+\lambda_k^2-2\lambda_j\lambda_k r_{jk}^{*}}{2(1-r_{jk}^{*2})}\right\}$$

$$\leqslant\sum_{k=m}^{n}\sum_{j=1}^{k-\gamma_k}\frac{\eta_{jk}\lambda_k\lambda_j}{(1-\eta_{jk}^2)^{1/2}}\psi(\lambda_j)\psi(\lambda_k)$$

$$\times\exp\left\{\frac{r_{jk}^{*2}(\lambda_j^2+\lambda_k^2)-2\lambda_j\lambda_k r_{jk}^{*}}{2(1-r_{jk}^{*2})}\right\}$$

$$\leqslant\sum_{k=m}^{n}\sum_{j=1}^{k-\gamma_k}\frac{\eta_{jk}\lambda_k^2}{(1-\eta_{jk}^2)^{1/2}}\psi(\lambda_j)\psi(\lambda_k)\exp\{\eta_{jk}\lambda_k^2\},$$

但当$j\leqslant k-\gamma_k$时,

$$\eta_{jk}\lambda_k^2\leqslant c\gamma_k^{\alpha-1}\lambda_k^2\leqslant c\lambda_k^{-2}\downarrow 0.$$

于是对任给的$\delta>0$，适当地选取$m$并固定之，我们所考察的第一个和

$$(1.5.8)\qquad I_1\leqslant\delta\sum_{k=m}^{n}\sum_{j=1}^{k-\gamma_k}P(A_k)P(A_j)\leqslant\delta\left(\sum_{k=1}^{n}P(A_k)\right)^2.$$

对于第二个和有

$$I_2\leqslant\sum_{k=m}^{n}\sum_{j=k-\gamma_k}^{k-s}\frac{r_{jk}\lambda_j}{\sqrt{2\pi}(1-r_{jk}^{*2})^{1/2}}\psi(\lambda_j)\exp\left\{-\frac{(\lambda_k-r_{jk}^{*}\lambda_j)^2}{2(1-r_{jk}^{*2})}\right\}$$

$$\leqslant\sum_{k=m}^{n}\sum_{j=k-\gamma_k}^{k-s}\frac{\lambda_j}{(1-r^2)^{1/2}}\psi(\lambda_j)\exp\left\{-\frac{\lambda_k^2}{2}\left(\frac{1-r}{1+r}\right)\right\},$$

其中$r=\max\{r_{jk}\}<1$.若记$B=(1-r)/(2(1+r))$,有

$$I_2\leqslant\sum_{k=m}^{n}\sum_{j=k-\gamma_k}^{k-s}\frac{\lambda_j\psi(\lambda_j)}{(1-r^2)^{1/2}}\exp(-B\lambda_k^2).$$

首先让我们考察指标 $k\in A=\{k: m\leqslant k\leqslant n, \lambda_k\geqslant((2/B)\log k)^{1/2}\}$ 上的和，我们有

$$(1.5.9)\quad \sum_{k\in A}\sum_{j=k-\gamma_k}^{k-1}\frac{\lambda_j\psi(\lambda_j)}{(1-r^2)^{1/2}}\exp(-B\lambda_k^2)$$

$$\leqslant c\sum_{k\in A}\frac{(\log k)^{1/2}}{k^2}\sum_{j=1}^{n}P(A_j)\leqslant c\sum_{j=1}^{n}P(A_j).$$

当 $k\in A'=\{k: m\leqslant k\leqslant n, \lambda_k<((2/B)\log k)^{1/2}\}$ 时，$\gamma_k<((2/B)\log k)^{2/(1-\alpha)}$ 且当 $j=k-\gamma_k$ 时，对某 $D>0$，$k<j+D(\log j)^{2/(1-\alpha)}=: j+\xi_j$，交换求和次序有

$$(1.5.10)\quad \sum_{k\in A'}\sum_{j=k-\gamma_k}^{k-1}\frac{\lambda_j\psi(\lambda_j)}{(1-r^2)^{1/2}}\exp(-B\lambda_k^2)$$

$$\leqslant c\sum_{j=m-\gamma_m}^{n-1}\sum_{k=j+1}^{j+\xi_j}\lambda_j\psi(\lambda_j)\exp(-B\lambda_k^2)$$

$$\leqslant c\sum_{j=m-\gamma_m}^{n-1}\xi_j\lambda_j\psi(\lambda_j)\exp(-B\lambda_j^2)\leqslant c\sum_{j=1}^{n}P(A_j).$$

这样，利用 (1.5.8)，(1.5.9)和(1.5.10)，对任给的 $\delta>0$，存在 $m$ 使得

$$I\leqslant\delta\left(\sum_{j=1}^{n}P(A_j)\right)^2+c\sum_{j=1}^{n}P(A_j),$$

这就足以证明在$\rho<1$时，(ii) 被满足.所以 (1.5.6) 成立，即 (1.5.2) 式被证明.

3°若条件 (iii)被满足，设$C(T)=\beta_T\sup\limits_{0\leqslant t\leqslant T-a_T}|X(t+a_T)-X(t)|$，我们来证

$$(1.5.11)\qquad \varliminf_{T\to\infty}C(T)\geqslant 1\qquad \text{a.s.}$$

对任给的$\delta>0$，取$T_n=(1+\delta)^n$，记$\zeta(T)=[Ta_T^{-1}]-1$，

$$V(k,n)=(X((k+1)a_{T_n})-X(ka_{T_n}))/\sigma(a_{T_n}),$$

其中$0\leqslant k\leqslant\zeta(T_n)$，$n\geqslant 1$.对一切$k$，$n$有$EV(k,n)=0$，$EV^2(k,n)$

$=1$，且可如2°中同样证明：若$k\geqslant j+1$，

$$r_n(k,j)=EV(k,n)V(j,n)\leqslant c(k-j)^{2(\alpha-1)}$$

且当$\alpha\leqslant 1/2$时是负的。

当$\alpha\leqslant 1/2$时，利用引理1.5.3，仿照定理1.1.1的证明，我们有

$$\sum_{n=1}^{\infty}P\{\max_{0\leqslant k\leqslant\zeta(T_n)}|V(k,n)|\leqslant\lambda_n\}<\infty.$$

当$\alpha>1/2$时，考虑一个来自引理1.5.3（ii）的外加的项。设$\varepsilon$充分小使$\varepsilon<1-\alpha$且$2(1-\varepsilon)^2>1+r$，其中$r=\max\{r_n(k,j):n\geqslant 1,\ 1\leqslant k<j\leqslant\zeta(T_n)\}$。那么

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\sum_{j=k+1}^{\zeta(T_n)}r_n(k,j)\phi(\lambda_n,\lambda_n;r_n^*(k,j))$$

$$=\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\left(\sum_{j=k+1}^{k+\mu_n-1}+\sum_{j=k+\mu_n}^{\zeta(T_n)}\right)\frac{r_n(k,j)}{(1-r_n^{*2}(k,j))^{1/2}}$$

$$\times\exp\left\{-\frac{\lambda_n^2}{1+r_n^*(k,j)}\right\},$$

其中$\mu_n=\lambda_n^{1/(1-\alpha)}$。设$\nu>0$，由$1+\nu=2(1-\varepsilon)^2/(1+r)$定义，那么第一个和

$$\leqslant c\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\sum_{j=k+1}^{k+\mu_n}\exp\{-(1+\nu)(\log T_n/a_{T_n}+\log\log T_n)\}$$

$$\leqslant c\sum_{n=1}^{\infty}\frac{T_n}{a_{T_n}}\mu_n\left(\frac{a_{T_n}}{T_n\log T_n}\right)^{1+\nu}<\infty.$$

第二个和不超过

$$c\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\sum_{j=k+\mu_n}^{\zeta(T_n)}\frac{1}{(j-k)^{2(1-\alpha)}}\exp\{-\lambda_n^2+c\lambda_n^2/\mu_n^{2(1-\alpha)}\}$$

$$\leqslant c\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\left(\frac{a_{T_n}}{T_n\log T_n}\right)^{2(1-\varepsilon)}\sum_{j=k+\mu_n}^{\zeta(T_n)}\frac{1}{(j-k)^{2(1-\alpha)}}$$

$$\leqslant c\sum_{n=1}^{\infty}\zeta^{2\alpha}(T_n)\left(\frac{a_{T_n}}{T_n\log T_n}\right)^{2(1-\varepsilon)}<\infty,$$

故由Borel-Cantelli引理，我们有

$$\varliminf_{n\to\infty} \lambda_n^{-1} \max_{0\leqslant k\leqslant \zeta(T_n)} |V(k,n)|\geqslant 1 \qquad \text{a.s.}$$

现在设$T_n\leqslant T<T_{n+1}$，那么$0\leqslant a_T-a_{T_n}\leqslant \delta a_T$，所以

$$\begin{aligned} C(T)&=\beta_T \sup_{0\leqslant t\leqslant T-a_T} |X(t+a_T)-X(t)| \\ &\geqslant \beta_{T_{n+1}} \max_{0\leqslant k\leqslant \zeta(T_n)} |V(k,n)|\sigma(a_{T_n}) \\ &\quad -\beta_T \sup_{0\leqslant t\leqslant T-\delta a_T} \sup_{0\leqslant s\leqslant \delta a_T} |X(t+s)-X(t)|, \end{aligned}$$

由1°和当$\delta$充分小时，$\beta_{T_{n+1}}/\beta_{T_n}$任意地接近于1，得证（1.5.11）成立.定理1.5.1证毕.

**注1.5.1** 洪圣岩（1990）讨论了Gauss过程$\{X(t)\}$的增量的下极限，得到与Book和Shore(1978)同样的结果.过程$\{X(t)\}$的连续模由陆传荣（1986）给出.

过程$\{X(t)\}$的滞后增量被陆传荣（1986）和洪圣岩（1990）所讨论.

**定理1.5.2** 设$\{X(t);t\geqslant 0\}$如上，满足（1.5.1）式，那么

(1.5.12) $\varlimsup_{T\to\infty} \sup_{0<t\leqslant T} |X(T)-X(T-t)|/d(T,t)=1$ a.s.

(1.5.13) $\lim_{T\to\infty} \sup_{0<t\leqslant T} \sup_{t\leqslant s\leqslant T} |X(s)-X(s-t)|/d(T,t)=1$ a.s.

(1.5.14) $\varlimsup_{T\to\infty} \sup_{0<t\leqslant T} \sup_{0\leqslant s\leqslant t} |X(T)-X(T-s)|/d(T,t)=1$ a.s.

其中$d(T,t)=\{2\sigma^2(t)(\log T/t+\log\log t)\}^{1/2}$.

证明类似于定理1.1.3和定理1.5.1，故在此省略.

**注1.5.2** 关于Gauss过程$\{X(t)\}$，对应于（1.1.44），（1.1.45），（1.1.72），（1.1.73）和（1.2.1），（1.2.2)的结果也是正确的.

### 1.5.2 较一般的Gauss过程的增量

设$\{\Gamma(t),t\geqslant 0\}$是具有平稳增量的Gauss过程，$E\Gamma(t)=0$.假设

(1.5.15) $\qquad \sigma^2(s)=E(\Gamma(t+s)-\Gamma(t))^2$

是$s$的单调不减函数，且$\sigma(s)=s^{\alpha}\sigma_1(s)$，$(s>0)$，其中$\alpha>0$，$\sigma_1(s)$也是s的不减函数.对这一类较一般的具有平稳增量的Gauss过程，在一定条件下也有相应于定理1.5.1的大增量结果.

**定理1.5.3** (Csaki等1990) 设$0<a_T\leqslant T$是$T$的函数，满足定理1.1.1的条件 (i) 和 (ii).若过程$\{\Gamma(t);\ t\geqslant 0\}$还满足条件：对任给$0\leqslant a<b\leqslant c<d$

$$E(\Gamma(d)-\Gamma(c))(\Gamma(b)-\Gamma(a))\leqslant 0. \tag{1.5.16}$$

那么我们有

$$\overline{\lim_{T\to\infty}}\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}\beta_T|\Gamma(t+s)-\Gamma(t)|=1\quad \text{a.s.} \tag{1.5.17}$$

$$\overline{\lim_{T\to\infty}}\sup_{0\leqslant t\leqslant T-a_T}\beta_T|\Gamma(t+a_T)-\Gamma(t)|=1\quad \text{a.s.} \tag{1.5.18}$$

$$\overline{\lim_{T\to\infty}}\beta_T|\Gamma(T+a_T)-\Gamma(T)|=1\quad \text{a.s.} \tag{1.5.19}$$

若还满足定理1.1.1的条件(iii)，那么在(1.5.17)和(1.5.18)中，$\overline{\lim}$可换成lim.

定理的证明类似于Csörgő和Révész (1981) 中定理 1.2.1的证明，其中第二步需应用拓广的Borel-Cantelli引理1.5.4.由于假设 (1.5.16) 成立，保证了这一可能。

# 第二章　独立随机变量部分和的增量

## § 2.1　引　　言

随机变量序列部分和的增量的几乎必然（a.s.）极限性质是概率论极限理论中一类十分深刻的结果. 对i.i.d.序列，借助Wiener过程增量的结果及用Wiener过程强逼近部分和过程的理论，人们已经获得了相当理想的结果.

设$\{X_n, n\geqslant 1\}$是i.i.d.随机变量序列，其共同的均值等于零，方差等于1. 又设$\{a_n\}$是一个单调不减的正整数序列，满足下列条件：

(i) $1\leqslant a_N\leqslant N$；

(ii) $N/a_N$是单调不减的.

记$S_n=\sum_{i=1}^{n}X_i$, $\beta_N=\{2a_N(\log(N/a_N)+\log\log N)\}^{-1/2}$. 首先，Csörgő和Révész（1981）考察了$\{S_n\}$的大增量性质，证明了

**定理2.1.1**　假设$\{X_n\}$满足条件

(2.1.1) 存在$t_0>0$，使对任意的$|t|\leqslant t_0$，成立$Ee^{tX_1}<\infty$.

又设$\{a_n\}$满足条件（i），（ii）和

(iii) $\lim\limits_{N\to\infty} a_N/\log N=\infty$.

那么

(2.1.2)
$$\overline{\lim_{N\to\infty}}\,\beta_N|S_{N+a_N}-S_N|=1\ \text{a.s.}$$

(2.1.3)
$$\overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N-a_N}\beta_N|S_{n+a_N}-S_n|=1\quad\text{a.s.}$$

(2.1.4)
$$\overline{\lim_{N\to\infty}}\max_{1\leqslant k\leqslant a_N}\beta_N|S_{N+k}-S_N|=1\quad\text{a.s.}$$

(2.1.5) $$\overline{\lim_{N\to\infty}} \max_{1\leq n\leq N-a_N} \max_{1\leq k\leq a_N} \beta_N |S_{n+k}-S_n| = 1 \quad \text{a.s.}$$

如果$\{a_n\}$还满足

(iv) $\lim\limits_{N\to\infty}(\log N/a_N)/\log\log N=\infty$,

那么

(2.1.6) $$\lim_{N\to\infty} \max_{1\leq n\leq N-a_N} \beta_N |S_{n+a_N}-S_n| = 1 \quad \text{a.s.}$$

(2.1.7) $$\lim_{N\to\infty} \max_{1\leq n\leq N-a_N} \max_{1\leq k\leq a_N} \beta_N |S_{n+k}-S_n| = 1 \quad \text{a.s.}$$

如果我们不是假设矩母函数存在，而仅仅假设有限阶矩的存在性，Csörgő和Révész(1981) 证明了

**定理2.1.2** 假设$H(x)$，$x>0$，是一个不减的连续函数，满足下列条件：

(2.1.8) $$EH(|X_1|)<\infty;$$

(2.1.9)对任意的$\varepsilon>0$，$\varliminf\limits_{x\to\infty} H(\varepsilon x)/H(x)>0$;

(2.1.10)对某个$\varepsilon>0$，$x^{-(2+\varepsilon)}H(x)$是$x$的增函数；

(2.1.11)$x^{-1}\log H(x)$是不增的。

附加于条件 (i) 和 (ii)，对$\{a_n\}$存在$C>0$使得

(2.1.12) $$a_N\geq C(\mathrm{inv}H(N))^2/\log N.$$

那么定理2.1.1的结论仍然成立。

继上述工作之后，Hanson 和 Russo (1983) 考虑了 i.i.d. 随机变量序列部分和的另一种形式的增量——滞后和。记$d(N,k)=\{2k[\log(N/k)+\log\log k]\}^{\frac{1}{2}}$。他们证明了

**定理2.1.3** 假设$\{X_n\}$满足定理2.1.1的条件。又设$\{a_n\}$满足条件 (i) 和 (iii)，那么我们有

(2.1.13) $$\overline{\lim_{N\to\infty}} \max_{a_N\leq k\leq N} |S_N-S_{N-k}|/d(N,k)=1 \quad \text{a.s.}$$

(2.1.14) $$\overline{\lim_{N\to\infty}} \max_{a_N\leq k\leq N} \max_{1\leq j\leq k} |S_N-S_{N-j}|/d(N,k)=1 \quad \text{a.s.}$$

(2.1.15) $\overline{\lim_{N\to\infty}}\max_{0\leqslant m\leqslant n\leqslant N}\max_{a_N\leqslant n-m}|S_n-S_m|/d(n,n-m)=1$ a.s.

(2.1.16) $\overline{\lim_{N\to\infty}}\max_{0\leqslant m\leqslant j\leqslant k\leqslant n\leqslant N}\max_{a_N\leqslant n-m}|S_k-S_j|/d(n,n-m)=1$

a.s.

如果条件 (2.1.1) 和 (iii) 分别用下列条件代替:

(2.1.17)存在$r>2$，使得$E|X_1|^r<\infty$,

(2.1.18) $\underline{\lim_{N\to\infty}}a_N(\log N)/N^{2/r}>0$,

那么定理的结论仍然成立。

Csörgő 和 Révész (1981) 还进一步考虑了 i.i.d.随机变量序列部分和增量有多小的问题. 通过应用Moguls'kii (1974) 的一个小偏差定理，他们给出了下列定理.

**定理2.1.4** 假设$\{X_n\}$是i.i.d.随机变量序列，均值为0，方差为1.又设$\{a_n\}$是一列不减的正整数，满足条件 (i), (ii) 和 (iii), 那么我们有

(2.1.19) $\underline{\lim_{N\to\infty}}\min_{1\leqslant n\leqslant N-a_N}\max_{1\leqslant k\leqslant a_N}\gamma_N|S_{n+k}-S_n|=1$ a.s.

其中$\gamma_N=\left(\frac{8}{\pi^2}\frac{\log N/a_N+\log\log N}{a_N}\right)^{1/2}$. 如果又附加条件 (iv),

那么

(2.1.20) $\lim_{N\to\infty}\min_{1\leqslant n\leqslant N-a_N}\max_{1\leqslant k\leqslant a_N}\gamma_N|S_{n+k}-S_n|=1$ a.s.

他们在书中并没有给出这个定理的详细证明，只是说道：这个定理可以通过重复Wiener过程的一个相应定理 (Csörgő, Révész 1981, 定理1.7.1) 的证明去论证.遗憾的是通过细致考察那个定理的证明 (检验 Csörgő和Révész (1981)中第49—50页上从 (1.7.4) 式到 (1.7.5) 式的证明),完全地重复 Wiener 过程时的证明是行不通的.

在这一章里，我们将要给出独立但未必同分布的随机变量序

列部分和的增量的a.s.极限性质.这时的麻烦在于还没有象i.i.d.情形时那样的十分理想的强逼近结果. 林正炎（1986a，1987，1988b）通过直接的途径得到了相应于定理2.1.1和2.1.2的结果.与此同时，加在$\{a_n\}$上的条件（ii）也本质上被减弱了.

Hanson和Russo（1985）推广他们自己的关于i.i.d.随机变量滞后和的结果到独立但不必同分布的情形. 但结论是不够理想的. 林正炎（1988a）改进了他们的主要结论，使之与i.i.d.情形时的理想情形相对应.

邵启满（1989）利用Skorohod嵌入方法进一步推广了林正炎的结果到更为一般的场合.

在第2.2和2.3两节，我们将叙述并证明上述结果.

上面提到的所有结论（无论是对i.i.d.情形还是非i.i.d.情形）都是在矩（或矩母函数）存在的条件下给出的. 但是，强极限定理（原则上）只依赖于概率而不依赖于矩. 林正炎（1990）讨论了不加矩假设的独立随机变量序列部分和的大增量问题. 证明的定理可以看作是带有矩条件的定理的一种推广.最近,林正炎和邵启满（1990）减弱了这个定理所要求的条件.

在第2.5节中，证明了一个邵启满（1989）得到的有关小增量的定理. 它不仅给出了定理2.1.4的正确证明，而且还把结果推广到了不同分布的情形.

Sakhanenko（1984）通过改进Komlós，Major和Tusnády（1975，1976）的方法，对独立非同分布随机变量建立了强逼近定理. 邵启满（1989）进一步改善Sakhanenko的结果，作为强逼近定理的推论还给出了独立非同分布随机变量序列的增量的某些结果. 在最后一节中，我们将叙述有关结果，它们不同于2.2及2.3节的结果.

## § 2.2 滞后和有多大?

研究滞后和的动机来自一类特殊的统计问题. 当人们利用一

个样本$X_1,\cdots,X_n$去估计总体的均值时，较早的$X_k$常可能存在较大的偏差。人们期望通过丢掉某些较早的$X_k$而去避免这种误差。Hanson和Russo(1983)通过证明Wiener过程相应增量的极限性质（参见第一章§1.1），对i.i.d.随机变量序列的滞后和首次获得了若干极限结果。尔后，他们(1985)将这些结果推广到独立但不必同分布的情形。关于i.i.d.序列，他们的结论是接近于理想的，但对非i.i.d.情形，就不然了。林正炎（1988a）改进了这些结果使之达到与i.i.d.情形时相对应的地步。

设$\{X_n,n\geqslant 1\}$是独立随机变量序列，$EX_n=0$ $(n\geqslant 1)$。记

$$S_n=\sum_{i=1}^{n}X_i,\sigma_n^2=EX_n^2,\ \sigma_{nk}^2=\sum_{i=n-k+1}^{n}\sigma_i^2 \text{ 和 } g(N,k)=\sigma_{Nk}$$

$\times\{2[\log(N/k)+\log\log k]\}^{\frac{1}{2}}$。

定理2.2.1（林正炎，1988a）假设$\{X_n\}$满足条件

(i) $$\liminf_{m\to\infty}\inf_{n\geqslant 0}\sum_{i=m+1}^{m+n}\sigma_i^2/n>0;$$

(ii) 存在$r>2$使对任意的$\varepsilon>0$,

$$\sum_{n=1}^{\infty}P(|X_n|^r>\varepsilon n)<\infty$$

且对任意的$s<r$和每一$n$,

$$E|X_n|^s\leqslant M<\infty.$$

那么对任意的$d>0$,

$$\overline{\lim_{N\to\infty}}\max_{dN^{2/r}/\log N\leqslant k\leqslant N}|S_N-S_{N-k}|/g(N,k)=1\quad \text{a.s.} \tag{2.2.1}$$

$$\overline{\lim_{N\to\infty}}\max_{dN^{2/r}/\log N\leqslant k\leqslant N}\max_{1\leqslant j\leqslant k}|S_N-S_{N-j}|/g(N,k)=1\quad \text{a.s.} \tag{2.2.2}$$

为了证明这个定理，我们需要下列引理。

引理2.2.1 (a) 假设存在正常数$g_1,\cdots,g_n$和$T$，使对$0\leqslant t\leqslant T$成立

$$Ee^{tX_k}\leqslant e^{g_k t^2/2}\quad k=1,2,\cdots,n.$$

记 $G=\sum_{k=1}^{n}g_k$. 那么

(2.2.3) $P\{\max\limits_{1\leqslant k\leqslant n}S_k\geqslant x\}\leqslant e^{-x^2/2G}$ 若$0<x\leqslant GT$,

(2.2.4) $P\{\max\limits_{1\leqslant k\leqslant n}S_k\geqslant x\}\leqslant e^{-Tx/2}$ 若$x\geqslant GT$.

(b) 假设存在某个$d>0$，使对$1\leqslant k\leqslant n$和$n\geqslant 1$,

$$|X_k|\leqslant ds_n,$$

其中 $s_n^2=\sum_{k=1}^{n}\sigma_k^2$. 那么对$\varepsilon>0$，存在$\gamma(\varepsilon)$和 $\pi(\varepsilon)$，当$\gamma$满足 $\gamma\geqslant\gamma(\varepsilon)$且$\gamma d\leqslant\pi(\varepsilon)$时成立

(2.2.5) $$P\{S_n\geqslant\gamma s_n\}\geqslant e^{-(1+\varepsilon)\gamma^2/2}.$$

结论 (a) 和(b)可以分别从Petrov(1975) 和Stout (1974) 中找到.

定理2.2.1的证明.

对$0<\varepsilon<\frac{1}{2}-\frac{1}{r}$，记 $\beta=\frac{1}{2}+\frac{1}{r}+\varepsilon$. 选取 $a>0$ 使得 $\eta_1=2a\left(r-2+\frac{1}{r}-a\right)<\frac{1}{2}-\frac{1}{r}$，又选$m$和$\beta_i$，$i=0,\cdots,m$，使得$\frac{2}{r}+\eta_1=\beta_0<\beta_1<\cdots<\beta_m=\beta$ 且对 $i=1,\cdots,m$，满足 $(1-\beta_i)/(1-\beta_{i-2})>\left(1+\frac{\varepsilon}{2}\right)^{-1}\left(\text{其中 }\beta_{-1}=\beta_0-\frac{\varepsilon}{4}\right)$. 易知 $\beta_i-\beta_{i-1}<\varepsilon/2$. 记$N(i)=[N^{\beta_i}]$. 写

$$\max_{aN^{2/r}/\log N\leqslant k\leqslant N^{\beta}}=\max_{aN^{2/r}/\log N\leqslant k\leqslant N^{\beta_0}}\vee\max_{1\leqslant i\leqslant m}\max_{N(i-1)\leqslant k\leqslant N(i)}.$$

对固定的$i$, $1\leqslant i\leqslant m$，考虑$\max\limits_{N(i-1)\leqslant k\leqslant N(i)}$.首先，我们来证明：存在$C_0>0$,

(2.2.6) $\overline{\lim\limits_{N\to\infty}}\max\limits_{N(t-1)\le k\le N(t)}\max\limits_{1\le j\le k}|S_N-S_{N-j}|/g(N,k)\le 1+C_0\varepsilon$

a.s.

记$Y_n=X_nI(|X_n|\le n^{\frac{1}{r}-\alpha})$，$Z_n=X_nI(n^{\frac{1}{r}-\alpha}<|X_n|\le n^{\frac{1}{r}})$，

$Y'_n=Y_n-EY_n$，$Z'_n=Z_n-EZ_n$，$T_n=\sum\limits_{i=1}^{n}Y'_i$，$U_n=\sum\limits_{i=1}^{n}Z'_i$，$\lambda_n^2=$

$\mathrm{Var}Y_n=\mathrm{Var}Y'_n$，$\lambda_{nk}^2=\sum\limits_{i=n-k+1}^{n}\lambda_i^2$. 由条件（ii）易知

$$\sum_{n=1}^{\infty}P(X_n\ne Y_n+Z_n)=\sum_{n=1}^{\infty}P(|X_n|>n^{\frac{1}{r}})<\infty.$$

因此$P\{X_n\ne Y_n+Z_n,\ \text{i.o.}\}=0$.从而

(2.2.7) 定理结论中的和$\Sigma X_i$可以用$\Sigma(Y_i+Z_i)$代替. 进一步，如取$s$使之满足 $(r+2)/2<s<r$，我们就有

$$|E(Y_i+Z_i)|=\left|\int_{|X_i|\le i^{1/r}}X_idP\right|\le i^{-(s-1)/r}E|X_i|^s<ci^{-1/2}.$$

由它推出：对$k\le N$，

$$\sum_{i=N-k+1}^{N}|E(Y_i+Z_i)|\le c\sum_{i=N-k+1}^{N}i^{-1/2}\le ck^{1/2}$$

$$=o(g(N,k)),N\to\infty.$$

所以 (2.2.7) 等价于

(2.2.8)定理结论中的和$\Sigma X_i$可以用$\Sigma(Y'_i+Z'_i)$ 代替.

我们来证明

(2.2.9) $\overline{\lim\limits_{N\to\infty}}\max\limits_{N(t-1)\le k\le N(t)}\max\limits_{1\le j\le k}|U_N-U_{N-j}|/g(N,k)\le\varepsilon$ a.s.

容易看出

$$EZ_n'^2=E|Z'_n|^{r-\alpha}|Z'_n|^{-(r-\alpha-2)}\le cn^{-(r-\alpha-2)(\frac{1}{r}-\alpha)}$$

$$=cn^{-1+\frac{2}{r}+\frac{\eta}{2}}.$$

因此，对$0<t\leqslant\frac{\eta}{2}n^{-1/r}\log n$ 和大的$n$有

$$(2.2.10)\quad Ee^{tZ_n'}=1+\frac{t^2}{2}EZ_n'^2+\frac{t^3}{6}EZ_n'^3+\frac{t^4}{24}EZ_n'^4+\cdots$$
$$\leqslant 1+\frac{t^2}{2}EZ_n'^2\left\{1+\frac{t}{3}2n^{1/r}+\frac{t^2}{12}2^2n^{2/r}+\cdots\right\}$$
$$\leqslant 1+\frac{t^2}{2}cn^{-1+\frac{2}{r}+\frac{\eta}{2}}\exp\left(\frac{\eta}{3}\log n\right)$$
$$\leqslant \exp\left\{\frac{t^2}{2}n^{-1+\frac{2}{r}+\eta}\right\}.$$

写

$$P_N:=P\{\max_{N(i-1)\leqslant k\leqslant N(i)}\max_{1\leqslant j\leqslant k}|U_N-U_{N-j}|/g(N,k)\geqslant\varepsilon\}$$
$$\leqslant P\{\max_{1\leqslant j\leqslant N(i)}|U_N-U_{N-j}|\geqslant\varepsilon\min_{N(i-1)\leqslant k\leqslant N(i)}g(N,k)\}.$$

令引理2.2.1中的参数$T=\frac{\eta}{2}N^{-1/r}\log N$，$g_k=N^{-1+2/r+\eta}$，$G=N(i-1)^{1/2}N^{1/r}$)容易验证$N(i)g_k\leqslant G$)，$x=\varepsilon\min\limits_{N(i-1)\leqslant k\leqslant N(i)}g(N,k)\leqslant c\varepsilon N(i-1)^{1/2}\log^{1/2}N$。对充分大的$N$，我们有 $0\leqslant x\leqslant GT$。于是，从（2.2.3）可得对充分对大的$N$

$$P_N\leqslant 2\exp\{-c\varepsilon^2N(i-1)^{1/2}N^{-1/r}\log N\}\leqslant N^{-2}.$$

（2.2.9）获证。所以由（2.2.8）推出

（2.2.11）　　定理结论中的和$\Sigma X_i$可以用$\Sigma Y_i'$代替。

显然，当$n\to\infty$时$\lambda_n/\sigma_n\to 1$。因此利用条件（i）和（ii），我们有

（2.2.12）　　当$n\to\infty$时，对$k$，$1\leqslant k\leqslant n$，一致地有 $\lambda_{nk}/\sigma_{nk}\to 1$。

于是，仿照（2.2.10），对$0<t\leqslant n^{-1/r+\alpha/2}$和大的$n$，

$$Ee^{tY_n'}\leqslant 1+\frac{t^2}{2}\lambda_n^2\exp\left\{\frac{2t}{3}n^{1/r-\alpha}\right\}\leqslant\exp\left\{\frac{t^2}{2}\lambda_n^2(1+\varepsilon)\right\}.$$

记$N'(i)=[(\beta_i-\beta_{i-1})\log N/\log B]+1$，其中$B>1$将在后面给定。

写

$$(2.2.13)\quad \max_{N(i-1)\leqslant k\leqslant N(i)} \max_{1\leqslant j\leqslant k} |T_N - T_{N-j}|/\lambda_{Nk}\{2[\log(N/k)$$

$$+\log\log k]\}^{1/2}$$

$$\leqslant \max_{1\leqslant l\leqslant N'(i)} \{ \max_{1\leqslant j\leqslant B^{l-1}N(i-1)} |T_N - T_{N-j}|$$

$$/\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^l N(i-1))^{1/2}$$

$$+\max_{B^{l-1}N(i-1)<j\leqslant B^l N(i-1)} |Y'_{N-j+1}+\cdots+Y'_{N-[B^{l-1}N(i-1)]}|$$

$$/\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^l N(i-1)))^{1/2}\}.$$

又记$n_k=n(k,i)=[k^{(1-\beta_{i-2})^{-1}}]$, $n_{k,i}=[(1-\beta_{i-2})^{-1}(n_k+1)^{\beta_{i-2}}]$，且令引理 2.2.1 中的参数 $T=n_k^{-1/r+a/2}$,

$$G=(1+\varepsilon)\sum_{j=n_k-n_{k,i}}^{n_k}\lambda_j^2,\quad x=(1+\varepsilon)\lambda_{n_k,n_{k,i}}(2\log n_k)^{1/2}.$$

我们也能够验证 $0\leqslant x\leqslant GT$。因此，由 (2.2.3)，我们有

$$P\{\max_{1\leqslant j\leqslant n_{k,i}} |T_{n_k} - T_{n_k-j}|\geqslant(1+\varepsilon)\lambda_{n_k,n_{k,i}}(2\log n_k)^{1/2}\}$$

$$\leqslant 2\exp\{-(1+\varepsilon)(1-\beta_{i-2})^{-1}\log k\}.$$

由此得

$$(2.2.14)\quad \varlimsup_{k\to\infty}\max_{1\leqslant j\leqslant n_{k,i}} |T_{n_k}-T_{n_k-j}|/\lambda_{n_k,n_{k,i}}(2\log n_k)^{1/2}$$

$$\leqslant 1+\varepsilon \qquad \text{a.s.}$$

类似地，令$T=n_k^{-1/r+a/2}$, $G=(1+\varepsilon)\sum_{j=n_k-n(k,i,l,\varepsilon)}^{n_k}\lambda_j^2$,

$x=(1+\varepsilon)\lambda_{n_k,n(k,i,l,\varepsilon)}(2\log(n_k/B^l n_k(i-1)))^{1/2}$，其中$n(k,i,l,\varepsilon)=[(1+\varepsilon)B^{l-1}n_k(i-1)]$。我们也有$0\leqslant x\leqslant GT$。注意到

$$n_k/B^l n_k(i-1)\geqslant n_k/n_k(i)\sim n_k^{1-\beta_i}\geqslant k^{(1+\varepsilon/2)^{-1}},$$

所以成立着

$$P\{\max_{1\leqslant l\leqslant n'_k(i)}\max_{1\leqslant j\leqslant n(k,i,l,\varepsilon)}|T_{n_k}-T_{n_k-j}|$$

$$\geqslant(1+\varepsilon)\lambda_{n_k,n(k,i,l,\varepsilon)}(2\log(n_k/B^l n_k(i-1)))^{1/2}\}$$

$$\leqslant \sum_{l=1}^{n_k^l(i)} 2\exp\left\{-(1+\varepsilon)\left(1+\frac{\varepsilon}{2}\right)^{-1}\log k\right\} \leqslant c(\log k)k^{-(1+\varepsilon/3)}.$$

从而

(2.2.15) $$\varlimsup_{k\to\infty} \max_{1\leqslant l\leqslant n_k^l(i)} \max_{1\leqslant j\leqslant n(k,i,l,\varepsilon)} |T_{n_k}-T_{n_k-j}|$$

$$/\lambda_{n_k,n(k,i,l,\varepsilon)}(2\log(n_k/B^l n_k(i-1)))^{1/2} \leqslant 1+\varepsilon \quad \text{a.s.}$$

对任意的正整数$N$，存在$k$，使得$n_{k-1}<N\leqslant n_k$。写

(2.2.16) $$\frac{|T_N-T_{N-j}|}{\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^l N(i-1)))^{1/2}}$$

$$\leqslant \frac{|T_{n_k}-T_N|}{\lambda_{n_k,n_{k,i}}(2\log n_k)^{1/2}} \frac{\lambda_{n_k,n_{k,i}}(2\log n_k)^{1/2}}{\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^l N(i-1)))^{1/2}}$$

$$+\frac{|T_{n_k}-T_{N-j}|}{\lambda_{n_k,n(k,i,l,\varepsilon)}(2\log(n_k/B^l n_k(i-1)))^{1/2}}$$

$$\times\frac{\lambda_{n_k,n(k,i,l,\varepsilon)}(2\log(n_k/B^l n_k(i-1)))^{1/2}}{\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^l N(i-1)))^{1/2}}$$

$$:=I_{11}\times I_{12}+I_{21}\times I_{22}.$$

利用中值定理我们得

$$n_k-N\leqslant n_k-n_{k-1}\leqslant(1-\beta_{i-2})^{-1}k^{(1-\beta_{i-2})^{-1}-1}$$

$$\leqslant(1-\beta_{i-2})^{-1}(n_k+1)^{\beta_{i-2}},$$

也就是说

(2.2.17) $$n_k-N\leqslant n_{k,i}.$$

由此从(2.2.14)推出$\varlimsup_{N\to\infty} I_{11}\leqslant 1+\varepsilon$ a.s. 利用（2.2.12）和条件(i)，(ii)，存在$0<C_1\leqslant C_2<\infty$使对每一$m, n$有

$$C_1 n\leqslant \sum_{i=m+1}^{n+m}\lambda_i^2\leqslant C_2 n.$$

那么由于当$N\to\infty$时

$$n_{k,i}/B^{l-1}N(i-1)\leqslant c n_k^{\beta_{i-2}}/N^{\beta_{i-1}}\to 0,$$

关于$l$，$1\leqslant l\leqslant N'(i)$，一致地有$\lim_{N\to\infty} I_{12}=0$。因而

$$(2.2.18)\qquad \lim_{N\to\infty}\max_{1\le l\le N'(i)} I_{11}\times I_{12}=0 \qquad \text{a.s.}$$

再来考虑$I_{21}$和$I_{22}$。对$1\le j\le B^{l-1}n_k(i-1)$，

$$n_k-N+j\le n_{k,i}+B^{l-1}n_k(i-1)\le n(k,i,l,\varepsilon).$$

所以对$I_{21}$，由（2.2.15）推出

$$(2.2.19)\qquad \overline{\lim_{N\to\infty}}\max_{1\le l\le N'(i)}\max_{1\le j\le B^{l-1}N(i-1)}|T_{n_k}-T_{N-j}|/$$

$$\lambda_{n_k,n(k,i,l,\varepsilon)}(2\log(n_k/B^l n_k(i-1)))^{1/2}\le 1+\varepsilon \quad \text{a.s.}$$

对$I_{22}$，记$q=(N-[B^{l-1}N(i-1)])\wedge(n_k-n(k,i,l,\varepsilon))$，$Q=(N-[B^{l-1}N(i-1)])\vee(n_k-n(k,i,l,\varepsilon))$。写

$$(2.2.20)\frac{\lambda^2_{n_k,n(k,i,l,\varepsilon)}}{\lambda^2_{N,[B^{l-1}N(i-1)]}}\le\frac{\lambda^2_{n_k,n_k-N}+\lambda^2_{N,[B^{l-1}N(i-1)]}+\lambda^2_{Q,Q-q}}{\lambda^2_{N,[B^{l-1}N(i-1)]}}.$$

从(2.2.17)，当 $N\to\infty$ 时 $\lambda^2_{n_k,n_k-N}/\lambda^2_{N,[B^{l-1}N(i-1)]}\to 0$，而$Q-q\le n_{k,i}+\varepsilon B^{l-1}{}_{n_k(i-1)}$。所以存在$C'>0$，使得

$$\overline{\lim_{N\to\infty}}\lambda^2_{Q,Q-q}/\lambda^2_{N,[B^{l-1}N(i-1)]}\le C'\varepsilon.$$

将这些结果插入(2.2.20)，就得到对$l$一致地成立

$$(2.2.21)\qquad \overline{\lim_{N\to\infty}}\lambda^2_{n_k,n(k,i,l,\varepsilon)}/\lambda^2_{N,[B^{l-1}N(i-1)]}\le 1+C'\varepsilon.$$

容易看出

$$(2.2.22)\qquad \lim_{N\to\infty}\log(n_k/B^l n_k(i-1))/\log(N/B^l N(i-1))=1.$$

结合（2.2.21）、(2.2.22）和（2.2.19）即得，存在$C_0>0$，

$$(2.2.23)\qquad \overline{\lim_{N\to\infty}}\max_{1\le l\le N'(i)}\max_{1\le j\le B^{l-1}N(i)} I_{21}\times I_{22}$$

$$\le(1+\varepsilon)(1+C'\varepsilon)^{1/2}\le 1+(C_0-1)\varepsilon \qquad \text{a.s.}$$

因此对（2.2.13）右边的第一部分，我们有

$$(2.2.24)\qquad \overline{\lim_{N\to\infty}}\max_{1\le l\le N'(i)}\max_{1\le j\le B^{l-1}N(i-1)}|T_N-T_{N-j}|/$$

$$\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^l N(i-1)))^{1/2}$$

$$\le 1+(C_0-1)\varepsilon \qquad \text{a.s.}$$

对（2.2.13）右边的第二部分，写

$$P\{\max_{1\leqslant i\leqslant N'(i)}\max_{B^{i-1}N(i-1)<j\leqslant B^iN(i)}|Y'_{N-j+1}+\cdots+Y'_{N-[B^{i-1}N(i-1)]}|/$$

$$\lambda_{N,[B^{i-1}N(i-1)]}(2\log(N/B^iN(i-1)))^{1/2}\geqslant\varepsilon\}$$

$$\leqslant\sum_{i=1}^{N'(i)}\sum_{j=[B^{i-1}N(i-1)]+1}^{[B^iN(i-1)]}P\{|Y'_{N-j+1}+\cdots+Y'_{N-[B^{i-1}N(i-1)]}|$$

$$\geqslant\varepsilon\lambda_{N,[B^{i-1}N(i-1)]}(2\log(N/B^iN(i-1)))^{1/2}\}.$$

估计式右边的概率．令$T=(N-[B^{i-1}N(i-1)])^{-1/r+\alpha/2}$，

$$G=\sum_{j=N-[B^iN(i-1)]+1}^{N-[B^{i-1}N(i-1)]}\lambda_j^2,$$

$$x=\varepsilon\lambda_{N,[B^{i-1}N(i-1)]}(2\log(N/B^iN(i-1)))^{1/2}.$$

我们也有 $0\leqslant x\leqslant GT$．于是从（2.2.3），只要$B=B(\varepsilon)$ 充分接近于 1，就有

$$P\{|Y'_{N-j+1}+\cdots+Y'_{N-[B^{i-1}N(i-1)]}|$$

$$\geqslant\varepsilon\lambda_{N,[B^{i-1}N(i-1)]}(2\log(N/B^iN(i-1)))^{1/2}\}$$

$$\leqslant 2\exp\left\{-\varepsilon^2\lambda^2_{N,[B^{i-1}N(i-1)]}\log(N/B^iN(i-1))\Big/\sum_{j=N-[B^iN(i-1)]+1}^{N-[B^{i-1}N(i-1)]}\lambda_j^2\right\}$$

$$\leqslant 2\exp\left\{-\varepsilon^2\frac{C_1(1-\beta)}{C_2(B-1)}\log N\right\}\leqslant N^{-3}.$$

因此

$$P\{\max_{1\leqslant i\leqslant N'(i)}\max_{B^{i-1}N(i-1)<j\leqslant B^iN(i-1)}|Y'_{N-j+1}+\cdots+Y'_{N-[B^{i-1}N(i-1)]}|/$$

$$\lambda_{N,[B^{i-1}N(i-1)]}(2\log(N/B^iN(i-1)))^{1/2}\geqslant\varepsilon\}$$

$$\leqslant N'(i)(B-1)B^{N'(i)-1}N(i-1)N^{-3}\leqslant cN^{-2}.$$

它推出

$$\varlimsup_{N\to\infty}\max_{1\leqslant i\leqslant N'(i)}\max_{B^{i-1}N(i-1)<j\leqslant B^iN(i-1)}|Y'_{N-j+1}+\cdots+Y'_{N-[B^{i-1}N(i-1)]}|/$$
$$\lambda_{N,[B^{i-1}N(i-1)]}(2\log(N/B^iN(i-1)))^{1/2}\leqslant\varepsilon \tag{2.2.25}$$

a.s.

结合（2.2.24）、（2.2.25）和（2.2.13）产生

(2.2.26) $$\overline{\lim_{N\to\infty}}\max_{(i-1)\leqslant k\leqslant N(l)}\max_{1\leqslant j\leqslant k}|T_N-T_{N-j}|/\lambda_{Nk}\{2[\log(N/k)+\log\log k]\}^{1/2}\leqslant 1+C_0\varepsilon$$

a.s.

回顾（2.2.11）和（2.2.12）得证（2.2.6）。

类似的方法可以证明

(2.2.27) $$\overline{\lim_{N\to\infty}}\max_{4N^{2/r}/\log N\leqslant k\leqslant N^{\beta_0}}\max_{1\leqslant j\leqslant k}|S_N-S_{N-j}|/g(N,k)\leqslant 1+\varepsilon \quad \text{a.s.}$$

（这时，要求$\sum_{n=1}^{\infty}P\{|X_n|^r>\varepsilon n\}<\infty$对任意的$\varepsilon>0$都满足）

我们略去了这一证明.由（2.2.6）和（2.2.27）产生

(2.2.28) $$\overline{\lim_{N\to\infty}}\max_{4N^{2/r}/\log N\leqslant k\leqslant N^{\beta}}\max_{1\leqslant j\leqslant k}|S_N-S_{N-j}|/g(N,k)\leqslant 1+(C_0+1)\varepsilon \quad \text{a.s.}$$

其次，我们来证明

(2.2.29) $$\overline{\lim_{N\to\infty}}\max_{N^{\beta}<k\leqslant N}\max_{1\leqslant j\leqslant k}|S_N-S_{N-j}|/g(N,k)\leqslant 1 \quad \text{a.s.}$$

选取$s$使得$\frac{2}{r}<\frac{2}{s}<\frac{2}{r}+\varepsilon$.由 Strassen(1965)中的定理4.4，存在一个概率空间，在其上存在一个$\{X_n\}$的像（我们仍记作$\{X_n\}$）和一个相关的Wiener过程$W$，使得

$$S_n=W(\sigma_{nn}^2)+o(\sigma_{nn}^{1/2+1/s}\log\sigma_{nn}^2) \quad \text{a.s.}$$

从条件（ii），我们有

$$S_n=W(\sigma_{nn}^2)+o(\sigma(n^{1/4+1/2s}\log n) \quad \text{a.s.}$$

因此

(2.2.30) $$\max_{N^{\beta}<k\leqslant N}\max_{1\leqslant j\leqslant k}|S_N-S_{N-j}|/g(N,k)$$

$$\leqslant\max_{N^{\beta}<k\leqslant N}\max_{1\leqslant j\leqslant k}\frac{W(\sigma_{NN}^2)-W(\sigma_{N-j,N-j}^2)}{g(N,k)}+o(N^{\frac{1}{4}+\frac{1}{2s}}/N^{\frac{\beta}{2}}) \quad \text{a.s.}$$

从定理1.1.3的（2.1.27）式得到

$$(2.2.31)\ \overline{\lim_{N\to\infty}}\max_{N^{\beta}<k\leqslant N}\max_{1\leqslant j\leqslant k}|W(\sigma_{NN}^{2})-W(\sigma_{N-k,N-k}^{2})|/d(\sigma_{NN}^{2},\sigma_{Nk}^{2})\leqslant 1 \qquad \text{a.s.}$$

其中 $d(t,a)=\{2a[\log(t/a)+\log\log a]\}^{1/2}$.由条件（i）和（ii），对$k$，$N^{\beta}<k\leqslant N$，一致地有

$$(2.2.32)\quad \frac{d(\sigma_{NN}^{2},\sigma_{Nk}^{2})}{g(N,k)}=\left\{\frac{\log(\sigma_{NN}^{2}/\sigma_{Nk}^{2})+\log\log\sigma_{Nk}^{2}}{\log(N/k)+\log\log k}\right\}^{1/2}\to 1,\qquad N\to\infty.$$

此外，从$\frac{1}{2}+\frac{1}{s}\leqslant\frac{1}{2}+\frac{1}{r}+\frac{\varepsilon}{2}<\beta$ 推出

$$(2.2.33)\quad o(N^{\frac{1}{s}+\frac{1}{2s}}/N^{\frac{\beta}{2}})=o(1).$$

综合（2.2.30）—（2.2.33）得证（2.2.29）.

为了完成定理的证明，只需验证

$$(2.2.34)\ \overline{\lim_{N\to\infty}}\max_{aN^{2/r}\log N\leqslant k\leqslant N}|S_N-S_{N-k}|/g(N,k)\geqslant 1-\varepsilon \qquad \text{a.s.}$$

回顾（2.2.11）和（2.2.12），（2.2.34）等价于

$$(2.2.35)\quad \overline{\lim_{N\to\infty}}\max_{aN^{2/r}\log N\leqslant k\leqslant N}|T_N-T_{N-k}|/\lambda_{Nk}\{2[\log(N/k)+\log\log k]\}^{1/2}\geqslant 1-\varepsilon \qquad \text{a.s.}$$

令$n_k=[k^{r/(r-2)}]$.显然，若能证明

$$(2.2.36)\quad \overline{\lim_{k\to\infty}}|T_{n_k}-T_{n_k-[n_k^{2/r}]-1}|\Big/\left\{\lambda_{n_k,[n_k^{2/r}]+1}\left(2\left(1-\frac{2}{r}\right)\times\log n_k\right)^{1/2}\right\}\geqslant 1-\varepsilon \qquad \text{a.s.}$$

就足以得到（2.2.35）了.利用中值定理，对充分大的$k$，我们有

$$n_k-n_{k-1}\geqslant\frac{r}{r-2}(k-1)^{2/(r-2)}-1>n_k^{2/r}+1,$$

也就是说$n_k-[n_k^{2/r}]-1>n_{k-1}$.我们将要利用引理 2.2.1的（ii）.对充分大的$k$和$j\leqslant n_k$，易知

$$|Y_j'|\leqslant 2n_k^{1/r-\alpha}\leqslant 2k^{(1-\alpha r)/(r-2)}.$$

令$\delta=-\alpha\gamma/(r-2)$.存在$C^*>0$，成立着

$$|Y'_i|\leqslant 2k^{\delta+1/(r-2)}\leqslant C^*k^\delta\lambda_{n_k},[n_k^{2/r}]+1.$$

所以若取引理2.2.1(ii)中的$d=C^*k^\delta$，$\gamma=(1-\varepsilon)\left\{2\left(1-\frac{2}{r}\right)\right.$ $\left.\times\log n_k\right\}^{1/2}$，我们有

$$\begin{aligned}&P\{|T_{n_k}-T_{n_k-[n_k^{2/r}]-1}|\\&\geqslant(1-\varepsilon)\lambda_{n_k,[n_k^{2/r}]+1}\left(2\left(1-\frac{2}{r}\right)\log n_k\right)^{1/2}\}\\&\geqslant\exp\left\{-(1+\varepsilon)(1-\varepsilon)^2\left(1-\frac{2}{r}\right)\log n_k\right\}\geqslant k^{-(1-\varepsilon)}.\end{aligned}$$

注意到事件的独立性，应用Borel-Cantelli 引理，(2.2.36)得证。这就完成了定理2.2.1的证明。

考虑矩母函数存在的情形。假设

(iii) 存在$t_0>0$和$B>0$使对一切$k$和$|t|\leqslant t_0$，$Ee^{tX_k}\leqslant B$。

又设$\varphi_n$是一个正整数序列。

**定理2.2.2** 假设条件 (i) 和 (iii) 被满足，又设$1\leqslant\varphi_n\leqslant n$且当$n\to\infty$时，$\varphi_n/\log n\to\infty$。那么

(2.2.37) $$\overline{\lim_{N\to\infty}}\max_{\varphi_N\leqslant k\leqslant N}|S_N-S_{N-k}|/g(N,k)=1\quad\text{a.s.}$$

(2.2.38) $$\overline{\lim_{N\to\infty}}\max_{\varphi_N\leqslant k\leqslant N}\max_{1\leqslant j\leqslant k}|S_N-S_{N-j}|/g(N,k)=1\quad\text{a.s.}$$

借助于定理2.2.1，为了证明这个定理，只须证明

$$\overline{\lim_{N\to\infty}}\max_{\varphi_N\leqslant k\leqslant N}\max_{1\leqslant j\leqslant k}|S_N-S_{N-j}|/g(N,k)\leqslant 1\quad\text{a.s.}$$

它的证明步骤与定理2.2.1的相类似，从略。

## § 2.3 Csörgő-Révész增量有多大?

Csörgő-Révész增量是最早被研究的一类增量形式。这一节的目的在于推广Csörgő和Révész (1981) 的关于i.i.d.序列的定

理2.1.1和2.1.2到独立但不必同分布的情形，同时减弱这两个定理所要求的条件．林正炎（1986a，1987和1988b）首先通过直接估计的途径得到了相应于i.i.d.情形的一系列结果；邵启满（1989）又进一步改进了这些结果．

设 $\{X_n, n\geqslant 1\}$ 是独立的、均值为 0 的随机变量序列．记

$S_n=\sum_{j=1}^{n}X_j$，$\sigma_n^2=EX_n^2$．又设 $\{a_n,\ n\geqslant 1\}$是一个正整数序列．记

$$\sigma_{nN}^2=\sum_{j=n+1}^{n+a_N}\sigma_j^2,\quad \beta_{nN}=\{2\sigma_{nN}^2(\log(N/a_N)+\log\log N)\}^{-\frac{1}{2}}.$$

**定理2.3.1** 假设$\{X_n\}$满足下列条件：

(i) $\varliminf\limits_{n\to\infty}\inf\limits_{m\geqslant 0}\sum\limits_{j=m+1}^{m+n}\sigma_j^2/n>0$ 且存在 $\alpha>0$，使得对每一 $n$，$E|X_n|^{2+\alpha}\leqslant M<\infty$；

(ii) 存在一个连续不减函数 $H(x), x\geqslant 0$，满足

(2.3.1) 对任意的$\varepsilon>0$，$\sum\limits_{n=1}^{\infty}P\{H(|X_n|)>\varepsilon n\}<\infty$，

$$\varliminf_{x\to\infty}H\left(\frac{1}{2}x\right)/H(x)>0. \tag{2.3.2}$$

又设$\{a_n\}$满足条件

(a) 存在$a>0$，使得$a(\mathrm{inv}H(n))^2/\log n\leqslant a_n\leqslant n$.

那么成立着

$$\varlimsup_{N\to\infty}\beta_{NN}|S_{N+a_N}-S_N|=1 \quad \text{a.s.} \tag{2.3.3}$$

$$\varlimsup_{N\to\infty}\max_{1\leqslant n\leqslant N}\beta_{nN}|S_{n+a_N}-S_n|=1 \quad \text{a.s.} \tag{2.3.4}$$

$$\varlimsup_{N\to\infty}\max_{1\leqslant k\leqslant a_N}\beta_{NN}|S_{N+k}-S_N|=1 \quad \text{a.s.} \tag{2.3.5}$$

$$\varlimsup_{N\to\infty}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\beta_{nN}|S_{n+k}-S_n|=1 \quad \text{a.s.} \tag{2.3.6}$$

如果$\{a_n\}$还满足条件

(b) $\lim\limits_{N\to\infty}\log(N/a_N)/\log\log N=\infty$,

那么

(2.3.7) $$\lim_{N\to\infty}\max_{1\leqslant n\leqslant N}\beta_{nN}|S_{n+a_N}-S_n|=1 \quad \text{a.s.}$$

(2.3.8) $$\lim_{N\to\infty}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\beta_{nN}|S_{n+k}-S_n|=1 \quad \text{a.s.}$$

为了证明这个定理，需要下列

**引理2.3.1** 设$X$是一均值为0的随机变量，$a>0$，$0<\alpha\leqslant1$。那么，对任意的$t\geqslant0$，

$$E\exp\{tXI(X\leqslant a)\}\leqslant\exp\left\{\frac{t^2}{2}EX^2+t^{2+\alpha}e^{2ta}E|X|^{2+\alpha}\right\}.$$

**证** $E\exp\{tXI(X\leqslant a)\}$

$$=1+tEXI(X\leqslant a)+\frac{t^2}{2}EX^2I(X\leqslant a)$$

$$+E\left\{\sum_{j=3}^{\infty}\frac{t^jX^j}{j!}I(X\leqslant a)\right\}$$

$$\leqslant1+\frac{t^2}{2}EX^2+e^{ta}\frac{(ta)^{1-\alpha}}{6}t^{2+\alpha}E|X|^{2+\alpha}$$

$$\leqslant\exp\left\{\frac{t^2}{2}EX^2+t^{2+\alpha}e^{2ta}E|X|^{2+\alpha}\right\}.$$

定理2.3.1的证明.

首先我们证明(2.3.3)—(2.3.6)。为此只需证明以下三式就够了：对任意的$0<\varepsilon<\frac{1}{4}$,

(2.3.9) $$\varlimsup_{N\to\infty}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\beta_{nN}(S_{n+k}-S_n)\leqslant1+2\varepsilon \quad \text{a.s.}$$

(2.3.10) $$\varlimsup_{N\to\infty}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\beta_{nN}(S_n-S_{n+k})\leqslant1+2\varepsilon \quad \text{a.s.}$$

和

(2.3.11) $$\varlimsup_{N\to\infty}\beta_{NN}(S_{N+a_N}-S_N)\geqslant1-2\varepsilon \quad \text{a.s.}$$

令$A$为一个正数，其值将在后面给定。定义

$$Y_n=X_nI(|X_n|\leqslant A)-EX_nI(|X_n|\leqslant A),$$

$$Y_n'=X_nI(|X_n|>A)-EX_nI(|X_n|>A),$$

$$Z_n=Y_n'I\left(Y_n'\leqslant\frac{2}{A}\operatorname{inv}H(n)\right),$$

$$U_n=\sum_{i=1}^{n}Y_i,\quad T_n=\sum_{i=1}^{n}Z_i.$$

由条件 (2.3.2)，存在常数$d>1$，使得对任意的$x>0$，

$$dH\left(x/2\right)\geqslant H(x),$$

从而

$$d^mH(2^{-m}x)\geqslant H(x).$$

后者推出

$$2^{-m}\operatorname{inv}H(n)\geqslant\operatorname{inv}H(d^{-m}n).$$

因此，存在$d_A>0$使得

$$\frac{1}{A}\operatorname{inv}H(n)\geqslant\operatorname{inv}H(d_An).\tag{2.3.12}$$

此外，对任意的$2^m\leqslant x<2^{m+1}$ $(m\geqslant 0)$，

$$\begin{aligned}H(x)&\leqslant d^{(\log x)/\log 2}H(2^{-m}x)\\&\leqslant x^{(\log d)/\log 2}dH(1).\end{aligned}$$

由此可得：存在正常数$D$，$G$和$\gamma$，使得对任意的$x>0$，

$$H(x)\leqslant D+Gx^{\gamma}.\tag{2.3.13}$$

利用$EX_n=0$和(2.3.13)可以推得当$n$充分大时有 $|EX_nI(|X_n|>A)|\leqslant A\leqslant\frac{1}{A}\operatorname{inv}H(n)$。由此及 (2.3.12) 我们得对充分大的$n$

$$\begin{aligned}P\left\{Y_n'>\frac{2}{A}\operatorname{inv}H(n)\right\}&\leqslant P\left\{|X_n|\geqslant\frac{1}{A}\cdot\operatorname{inv}H(n)\right\}\\&\leqslant P\{H(|X_n|)\leqslant d_An\}.\end{aligned}$$

因此，从条件 (2.3.1) 得

(2.3.14) $$P\left\{Y_n' > \frac{2}{A}\,\mathrm{inv}H(n),\ \text{i.o.}\right\} = 0.$$

这样一来，为了证明(2.3.9)，只需证明以下两式就够了，

(2.3.15) $$\varlimsup_{N\to\infty}\max_{1\leq n\leq N}\max_{1\leq k\leq a_N}\beta_{nN}|U_{n+k}-U_n|\leq 1+\varepsilon \quad \text{a.s.}$$

(2.3.16) $$\varlimsup_{N\to\infty}\max_{1\leq n\leq N}\max_{1\leq k\leq a_N}\beta_{nN}(T_{n+k}-T_n)\leq\varepsilon \quad \text{a.s.}$$

先来证明(2.3.16)．由条件 (i)，存在正常数$\sigma$和$\sigma'$，使得对一切$n\geq 1$和充分大的$m$，

(2.3.17) $$\sigma^2 n\leq\sum_{i=m+1}^{m+n}\sigma_i^2\leq\sigma'^2 n.$$

记$A_k=\{n;\ 2^k\leq a_n<2^{k+1}\}$和$M_k=\max\{n;\ n\in A_k\}$．我们有

(2.3.18) $$\max_{N\geq L}\max_{1\leq n\leq N}\max_{1\leq k\leq a_N}\beta_{nN}(T_{n+k}-T_n)$$
$$\leq\max_{l\geq\log_2 a_L}\max_{N\in A_l}\max_{1\leq n\leq N}\max_{1\leq k\leq a_N}\beta_{nN}(T_{n+k}-T_n)$$
$$\leq\max_{l\geq\log_2 a_L}\max_{1\leq n\leq M_l}\max_{1\leq k\leq 2^{l+1}}(T_{n+k}-T_n)/$$
$$\sigma\{2^{l+1}\log(n(\log 2^l)/2^l)\}^{1/2}.$$

应用Lévy极大不等式和引理2.3.1，对一切大的$l$和任意的$t\geq 0$，

(2.3.19) $$P\{\max_{1\leq n\leq M_l}\max_{1\leq k\leq 2^{l+1}}(T_{n+k}-T_n)/$$
$$\sigma\{2^{l+1}\log(n(\log 2^l)/2^l)\}^{1/2}\geq\varepsilon\}$$
$$\leq\sum_{j=0}^{[M_l/2^l]+1}P\left\{\max_{1\leq k\leq 2^{l+1}}(T_{j2^l+k}-T_{j2^l})/\right.$$
$$\left.\sigma(2^{l+1}\log((j+1)\log 2^l))^{1/2}\geq\frac{\varepsilon}{4}\right\}$$
$$\leq 2\sum_{j=0}^{[M_l/2^l]+1}P\left\{(T_{(j+2)2^l}-T_{j2^l})/\right.$$
$$\left.\sigma(2^{l+1}\log((j+1)\log 2^l))^{1/2}\geq\frac{\varepsilon}{8}\right\}$$

$$\leqslant 2\sum_{j=0}^{[M_i/2^i]+1}\exp\Big\{-\frac{\varepsilon}{8}t\sigma(2^{i+1}\log((j+1)\log 2^i))^{1/2}$$

$$+\sum_{l=j2^i+1}^{(j+2)2^i}\Big(\frac{t^2}{2}EZ_l^2+t^{2+a}E|Z_l|^{2+a}e^{4t\,\mathrm{inv}H((j+2)2^i)/A}\Big)\Big\},$$

由条件 (i)，其中的

$$\sum_{l=j2^i+1}^{(j+2)2^i}EZ_l^2\leqslant 2\sum_{l=j2^i+1}^{(j+2)2^i}E|X_l|^{2+a}/A^a\leqslant 4\cdot 2^iM/A^a.$$

取$t=t_{ji}=\frac{32}{\sigma\varepsilon}\Big(\frac{\log((j+1)\log 2^i)}{2^i}\Big)^{1/2}$.因为从条件(a)和(2.3.13)式有

$$\max_{j\leqslant M_i/2^i+2}\frac{\log(j\log 2^i)}{2^i}\leqslant\max_{j\leqslant M_i/2^i+2}\frac{2\log(M_i\log M_i)}{a_{M_i}}\to 0,\quad i\to\infty,$$

所以

$$\max_{j\leqslant M_i/2^i+1}t_{ji}\to 0,\quad i\to\infty.$$

取$A$充分大，使其满足

$$A\geqslant(2M(32/\sigma\varepsilon)^2)^{1/a}$$

及对充分大的$i$

$$2t_{ji}^aM\exp\{4t_{ji}\mathrm{inv}H((j+2)2^i)/A\}\leqslant(\sigma\varepsilon/32)^2.$$

后者是可能的，因为由(2.3.13)和条件 (a)，存在$C'>0$，使得对$0\leqslant j\leqslant M_i/2^i+1$,

$$\frac{\log((j+1)\log 2^i)}{2^i}(\mathrm{inv}H((j+1)2^i))^2$$

$$\leqslant C'\Big(\log\frac{2^i}{\log((j+1)\log 2^i)}\Big)^2.$$

因此，从 (2.3.19) 得

$$P\{\max_{1\leqslant n\leqslant M_i}\max_{1\leqslant k\leqslant 2^{i+1}}(T_{n+k}-T_n)/\sigma\{2^i\log(n(\log 2^i)/2^i)\}^{1/2}$$

$$\geqslant\varepsilon\}$$

$$\leqslant c\sum_{j=0}^{[M_i/2^i]+1}\exp\{-2\log((j+1)\log 2^i)\}.$$

由此并注意到 (2.3.18) 即得证 (2.3.16).

为了证明(2.3.15), 修改 $A_k$的定义如下: $A_k=\{n:\ \theta^k\leqslant a_n<\theta^{k+1}\}$, 其中$\theta=\theta(\varepsilon)>1$在后面给定. $M_k$的 定义 如前. 这样, 类似于 (2.3.18) 有

$$\max_{N\geqslant L}\ \max_{1\leqslant n\leqslant N}\ \max_{1\leqslant k\leqslant a_N}\ \beta_{nN}|U_{n+k}-U_n|$$

$$\leqslant\max_{i\geqslant\log_\theta a_L}\ \max_{1\leqslant n\leqslant M_i}\ \max_{1\leqslant k\leqslant\theta^{i+1}}|U_{n+k}-U_n|/\sigma^*_{ni}$$

$$\times\{2\log((n\vee\theta^{i+1})(\log\theta^i)/\theta^{i+1})\}^{1/2},$$

其中 $\sigma^{*2}_{ni}=\sum_{j=n+1}^{n+[\theta^i]}\sigma_j^2$. 所以, (2.3.15) 可从下式推出:

$$\overline{\lim_{i\to\infty}}\ \max_{1\leqslant n\leqslant M_i}\ \max_{1\leqslant k\leqslant\theta^{i+1}}|U_{n+k}-U_n|/\sigma^*_{ni}\tag{2.3.20}$$

$$\times\{2\log((n\vee\theta^{i+1})(\log\theta^i)/\theta^{i+1})\}^{1/2}\leqslant 1+\varepsilon\quad\text{a.s.}$$

令 $r=r(\varepsilon)$充分大. 记 $R=[\theta^{i+1}/2^r]$, $n_r=[n/R]\ R$. 可写

(2.3.21) $|U_{n+k}-U_n|$

$$\leqslant|U_{(n+k)_r}-U_{n_r}|+|U_{n+k}-U_{(n+k)_r}|+|U_n-U_{n_r}|.$$

记$\sigma^{*2}_{ni}(r)=\sum_{j=n+1}^{n+[\theta^{i+1}]+R}\sigma_j^2$. 由(2.3.17), 对充分大的 $i$, 只要$r=r(\varepsilon)$足够大, $\theta=\theta(\varepsilon)$ 充分接近于 1, 就有

$$1\leqslant\sigma^*_{n,i}(r)/\sigma^*_{ni}\leqslant 1+\frac{\varepsilon}{10}.$$

利用Kolmogorov不等式, 我们得

$$P\Big\{\max_{1\leqslant n\leqslant M_i}\ \max_{1\leqslant k\leqslant\theta^{i+1}}|U_{(n+k)_r}-U_{n_r}|/\sigma^*_{ni}$$

$$\times\{2\log(n(\log\theta^i)/\theta^{i+1})\}^{1/2}\geqslant 1+\frac{\varepsilon}{3}\Big\}$$

$$\leqslant P\left\{\max_{0\leqslant j\leqslant M_i/R}\ \max_{1\leqslant k\leqslant \theta^{i+1}+R}|U_{jR+k}-U_{jR}|/\sigma^*_{jR+k,i}(r)\right.$$

$$\left.\times\left(2\log\frac{(jR+1)\log\theta^i}{\theta^{i+1}}\right)^{1/2}\geqslant 1+\frac{\varepsilon}{10}\right\}$$

$$\leqslant\sum_{j=i}^{[M_i/R]}2\exp\left\{-\left(1+\frac{\varepsilon}{10}\right)\log\frac{(jR+1)\log\theta^i}{\theta^{i+1}}\right\}$$

$$\leqslant ci^{-(1+\varepsilon/10)}.$$

由此推出

$$\overline{\lim_{i\to\infty}}\ \max_{1\leqslant n\leqslant M_i}\ \max_{1\leqslant k\leqslant\theta^{i+1}}|U_{(n+k)_r}-U_{n_r}|/\sigma^*_{ni}$$

$$\times\{2\log(n(\log\theta^i)/\theta^{i+1})\}^{1/2}\leqslant 1+\frac{\varepsilon}{3}\qquad\text{a.s.}$$

对于（2.3.21）右边的第二项，因由（2.3.17），对充分大的$r$有

$$\sigma^{*2}_{ni}/\max_{1\leqslant k\leqslant\theta^{i+1}}\sum_{l=(n+k)_r+1}^{(n+k)_r+R}\sigma_l^2\geqslant\sigma^2[\theta^i]/(\sigma'^2\theta^{i+1}/2^r)\geqslant 3b/\varepsilon^2,$$

所以，只要$i$充分大，成立着

$$P\left\{\max_{1\leqslant n\leqslant M_i}\ \max_{1\leqslant k\leqslant\theta^{i+1}}|U_{n+k}-U_{(n+k)_r}|/\sigma^*_{ni}\right.$$

$$\left.\times\{2\log((n\vee\theta^{i+1})(\log\theta^i)/\theta^{i+1})\}^{1/2}\geqslant\frac{\varepsilon}{3}\right\}$$

$$\leqslant\sum_{j=0}^{[(M_i+\theta^{i+1})/R]}P\left\{\max_{jR<l\leqslant(j+1)R}|U_l-U_{jR}|\Big/\right.$$

$$\left.\left(2\left(\sum_{l=jR+1}^{(j+1)R}\sigma_l^2\right)\log\frac{((jR-\theta^{i+1})\vee\theta^{i+1})\log\theta^i}{\theta^{i+1}}\right)^{1/2}\geqslant 2\right\}$$

$$\leqslant\sum_{j=0}^{2^{r+1}}2\exp\{-3\log\log\theta^i\}$$

$$+\sum_{j=2^{r+1}+1}^{\infty}2\exp\{-2\log((j/2^r-1)\log\theta^i)\}$$

$$\leqslant i^{-3}2^{r+2}\log\theta+2\log\theta\sum_{j=2^{r+1}+1}^{\infty}(i(j/2^r-1))^{-2}$$

$$\leqslant i^{-2}C(r)\log\theta,$$

其中$C(r)$是常数。因此我们得

$$\overline{\lim_{i\to\infty}}\max_{1\leqslant n\leqslant N_i}\max_{1\leqslant k\leqslant\theta^{i+1}}|U_{n+k}-U_{(n+k),r}|/\sigma^*_{ni}$$

$$\times\{2\log(n(\log\theta^i)/\theta^{i+1})\}^{1/2}\leqslant\frac{\varepsilon}{3}\qquad\text{a.s.}$$

对于(2.3.21)右边的第三项有相同的结论。综合这些结论即得(2.3.20)。这就证明了(2.3.15)。从(2.3.15)和(2.3.16)推出(2.3.9)。(2.3.10)的证明是完全类似的。

下面我们来证明(2.3.11)。由(2.3.14)和(2.3.16)，只需证明下式就足够了：

$$\overline{\lim_{N\to\infty}}\beta_{NN}(U_{N+a_N}-U_N)\geqslant 1-\varepsilon\qquad\text{a.s.}\tag{2.3.22}$$

记$N_1=1$。对$k\geqslant 2$，定义$N_k=N_{k-1}+a_{N_{k-1}}$。注意到独立性，(2.3.22)可从下式推得：

$$\sum_{k=1}^{\infty}P\{\beta_{N_kN_k}(U_{N_k+a_{N_k}}-U_{N_k})\geqslant 1-\varepsilon\}=\infty.\tag{2.3.23}$$

事实上，

$$\sum_{k=1}^{\infty}P\{\beta_{N_kN_k}(U_{N_k+a_{N_k}}-U_{N_k})\geqslant 1-\varepsilon\}$$

$$\geqslant\sum_{k=1}^{\infty}\exp\{-(1-\varepsilon)\log(N_k(\log N_k)/a_{N_k})\}$$

$$\geqslant\sum_{k=1}^{\infty}\frac{N_{k+1}-N_k}{N_k\log N_k}\geqslant\sum_{k=1}^{\infty}\int_{N_k}^{N_{k+1}}\frac{1}{x\log x}dx=\infty,$$

也就是说(2.3.23)成立。(2.3.11)得证。这就完成了定理的证明。

显然，这个定理在很大程度上推广和改进了定理2.1.1。相应

于定理2.1.2，我们有下列

**定理2.3.2** 假设$\{X_n\}$满足条件 (i) 和

(ii)′存在一个连续不减函数$H(x)$，$x\geqslant 0$，满足

(2.3.24) 存在$b>0$，使得$\sum\limits_{n=1}^{\infty} P\{H(|X_n|)>bn\}<\infty$；

(2.3.25) $x/\log H(x)$是不减的；

(2.3.26) 存在$\beta>0$，使得$E(H(|X_n|))^{\beta}\leqslant M<\infty$。

又设$\{a_n\}$满足条件

(a)′存在序列$b_n\uparrow\infty$，使得

$$b_n(\mathrm{inv}H(n))^2/\log n\leqslant a_n\leqslant n.$$

那么定理2.3.1的结论 (2.3.3)—(2.3.6)仍然成立.如果再附加定理2.3.1中的条件 (b)，那么(2.3.7)和 (2.3.8) 也是正确的.

这个定理可通过十分类似于定理2.3.1的证法去证明。需要指出的主要不同在于我们用了下列引理代替引理2.3.1.

**引理2.3.2** 设$X$是一均值为0的随机变量，$a>0$，$0<\alpha\leqslant 1$.假设$x/\log H(x)(x>0)$是不减的，那么，对$0\leqslant ta\leqslant \frac{a^2}{10}\log H(a)$，成立着

$$\begin{aligned}&E\exp\{tXI(X\leqslant a)\}\\&\leqslant\exp\left\{\frac{t^2}{2}EX^2+t^{2+\alpha/2}(E|X|^{2+\alpha})^{(4+\alpha)/(4+2\alpha)}\right.\\&\qquad\left.\times\ (E(H(|X|))^{\alpha})^{\alpha/(4+4\alpha)}\right\}.\end{aligned}$$

这个引理的证明与引理2.3.1完全相仿，故从略.

林正炎 (1990b) 还讨论了条件(b) 不成立时的下极限.

**定理2.3.3** (林正炎，1990b) 假设独立随机变量序列$\{X_n,n\geqslant 1\}$满足条件：

(i) $\varliminf\limits_{n\to\infty}\inf\limits_{m>0}E(X_{m+1}+\cdots+X_{m+n})^2/n>0$；

(ii) 存在$t_0>0$和$b>0$，使对一切$k$当$|t|\leqslant t_0$时，$Ee^{tX_k}\leqslant b$.

又设$\{a_n\}$是取正整值的不减常数序列，满足条件：

(a) $a_n \leqslant n$且$a_n/\log n \to \infty$, $(n\to\infty)$;

(b) $n/a_n$是单调不减的;

(c) $(\log(n/a_n))/\log\log\log n \to \infty$ $(n\to\infty)$.

那么我们有

(2.3.27) $\displaystyle \lim_{N\to\infty} \max_{0\leqslant n\leqslant N-a_N} \gamma_{nN}|S_{n+a_N}-S_n|=1$ a.s.

(2.3.28) $\displaystyle \lim_{N\to\infty} \max_{0\leqslant n\leqslant N-a_N} \max_{1\leqslant k\leqslant a_N} \gamma_{nN}|S_{n+k}-S_n|=1$ a.s.

其中 $\gamma_{nN}=\{2\sigma_{nN}^2\log(N/\sigma_{nN}^2\log\log N)\}^{-1/2}$.

**证** 由条件 (i), (ii) 易知, 对充分大的$N$存在$0<c_1\leqslant c_2<\infty$, 使得

(2.3.29) $$c_1 a_N \leqslant \sigma_{nN}^2 \leqslant c_2 a_N.$$

首先我们来证

(2.3.30) $\displaystyle \varliminf_{N\to\infty} \max_{0\leqslant n\leqslant N-a_N} \gamma_{nN}|S_{n+a_N}-S_n|\geqslant 1$ a.s.

记

$$Y_n=X_nI\left(|X_n|<A\left(\log\frac{n\log n}{a_n}\right)\Big/\left(\log\frac{n}{a_n\log\log n}\right)\right),$$

其中$A$是正的常数. 当

$$\varlimsup_{n\to\infty}\left(\log\frac{n\log n}{a_n}\right)\Big/\left(\log\frac{n}{a_n\log\log n}\right)<\infty$$

时, $A=A(\varepsilon)$ (对任意给定的$\varepsilon>0$) 取得充分大. 又记

$$Z_n=X_n-Y_n,\quad Y_n'=Y_n-EY_n,\quad Z_n'=Z_n-EZ_n,$$

$$U_n=\sum_{k=1}^n Y_k',\quad V_n=\sum_{k=1}^n Z_k',\ \lambda_k^2=\operatorname{Var} Y_k, \lambda_{nN}^2=\lambda_{n+1}^2+\cdots$$

$$+\lambda_{n+a_N}^2, \gamma_{nN}'=\{2\lambda_{nN}^2\log(N/\lambda_{nN}^2\log\log N)\}^{-1/2}.$$

由(2.3.29)可知对充分大$N$存在$0<c_3\leqslant c_4<\infty$使

(2.3.31) $$c_3 a_N \leqslant \lambda_{nN} \leqslant c_4 a_N.$$

易知对任意的$\varepsilon>0$, 对$n$一致地成立

(2.3.32) $$\varlimsup_{N\to\infty} |\gamma_{nN}/\gamma_{nN}'-1|\leqslant\varepsilon$$

（这里$A=A(\varepsilon)$ 取得充分大，当

$$\lim_{n\to\infty}\left(\log\frac{n\log n}{a_n}\right)\Big/\left(\log\frac{n}{a_n\log\log n}\right)=\infty$$

时，$\varepsilon$可取为0）。可以证明

(2.3.33) $\overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N-a_N}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|V_{n+k}-V_n|=0$ a.s.

事实上，对$|t|\leqslant t_0/2$，成立着

$$Ee^{tZ_n'}\leqslant\exp\left\{\frac{t^2}{24}\quad\varepsilon^2c_3\frac{\log(n/a_n\log\log n)}{\log((n\log n)/a_n)}\right\}.$$

注意到在条件（a）下，对充分大的$N$，

$$\frac{a_N\log(N/a_N\log\log N)}{\log^2((N\log N)/a_N)}$$

$$\geqslant\begin{cases}\dfrac{a_N}{\log N}\quad\dfrac{\log(N/(\log^2N)\log\log N)}{\log N}, & a_N<\log^2N,\\ \log(N/a_N\log\log N), & a_N\geqslant\log^2N.\end{cases}$$

故由条件（a）和（c）有

(2.3.34) $$\frac{a_N\log(N/a_N\log\log N)}{\log^2((N\log N)/a_N)}\to\infty,\quad N\to\infty.$$

这样，利用引理2.2.1即得(2.3.33)式。回顾(2.3.32)，(2.3.30)等价于

(2.3.35) $\underline{\lim}_{N\to\infty}\max_{0\leqslant n\leqslant N-a_N}\gamma_{nN}'|U_{n+a_N}-U_n|\geqslant1$ a.s.

写

(2.3.36) $P\{\max_{0\leqslant n\leqslant N-a_N}\gamma_{nN}'|U_{n+a_N}-U_n|<1-\varepsilon\}$

$$\leqslant\prod_{j=0}^{[N/a_N]-1}\left\{1-P\{\gamma_{ja_N,N}'|U_{(j+1)a_N}-U_{ja_N}|\geqslant1-\varepsilon\}\right\}.$$

由(2.3.34)可对序列$\{Y_n'\}$应用引理2.2.1，得

(2.3.37) $P\{\gamma_{ja_N,N}'|U_{(j+1)a_N}-U_{ja_N}|\geqslant1-\varepsilon\}$

$$\geqslant\exp\{-(1+\varepsilon)(1-\varepsilon)^2\log(N/\lambda_{ja_N,N}^2\log\log N)\}$$

$$\geqslant(\lambda_{i a_N, N}^{*}(\log\log N)/N)^{1-\varepsilon}\geqslant c\left(\frac{a_N\log\log N}{N}\right)^{1-\varepsilon},$$

将它代入(2.3.36)式并利用条件（c），对充分大的$N$，

(2.3.38) $$P\{\max_{0\leqslant n\leqslant N-a_N}\gamma'_{nN}|U_{n+a_N}-U_n|<1-\varepsilon\}$$

$$\leqslant\prod_{j=0}^{[N/a_N]-1}\left(1-c\left(\frac{a_N\log\log N}{N}\right)^{1-\varepsilon}\right)$$

$$\leqslant\exp\left\{-c\left(\frac{N}{a_N}\right)^{\varepsilon}(\log\log N)^{1-\varepsilon}\right\}\leqslant\log^{-3}N.$$

记 $N_k=2^{\sqrt{k}}$。由上式推得

(2.3.39) $$\varliminf_{k\to\infty}\max_{0\leqslant n\leqslant N_k-a_{N_k}}\gamma'_{nN_k}|U_{n+a_{N_k}}-U_n|\geqslant1\quad \text{a.s.}$$

对任给的正整数$N$，存在$k$使得$N_k\leqslant N\leqslant N_{k+1}$。写

(2.3.40) $$\max_{0\leqslant n\leqslant N-a_N}\gamma'_{nN}|U_{n+a_N}-U_n|$$

$$\geqslant\max_{0\leqslant n\leqslant N_k-a_{N_k}}\gamma'_{nN_k}|U_{n+a_{N_k}}-U_n|$$

$$-\max_{1\leqslant i\leqslant N_k}\max_{1\leqslant j\leqslant a_{N_{k+1}}-a_{N_k}}\gamma'_{nN_{k+1}}|U_{i+j}-U_i|$$

$$=:I_1(k)-I_2(k).$$

由条件（i），（ii)易知当$k\to\infty$时对$n$一致地成立

$$\lambda_{nN_k}/\lambda_{nN_{k+1}}\to1.$$

据此并结合（2.3.39）得

(2.3.41) $$\varliminf_{k\to\infty}I_1(k)\geqslant1\qquad \text{a.s.}$$

再来估计$I_2(k)$。记

$$M_k=N_k+a_{N_{k+1}}-a_{N_k},\quad \lambda_{ik}'^{2}=\lambda_{i+1}^2+\cdots+\lambda_{i+a_{N_{k+1}}-a_{N_k}}^2.$$

由定理2.3.1易知

(2.3.42) $$\varlimsup_{k\to\infty}\max_{1\leqslant i\leqslant N_k}\max_{1\leqslant j\leqslant a_{N_{k+1}}-a_{N_k}}|U_{i+j}-U_i|/(\lambda'_{ik}\{2(\log_k/$$

$$(M_k/\lambda_{ik}'^{2})+\log\log M_k)\}^{1/2})\leqslant1\qquad \text{a.s.}$$

由关于$\{a_n\}$的条件容易验证，当$k\to\infty$时对$n$和$i$一致地成立

$$\gamma'^{2}_{n_{N_{k+1}}}\{\lambda'^{2}_{ik}(\log(M_k/\lambda'^{2}_{ik})+\log\log M_k)\}$$

$$\leqslant c\,\frac{a_{N_{k+1}}-a_{N_k}}{a_{N_{k+1}}}\,\frac{\log(N_k/(a_{N_{k+1}}-a_{N_k}))+\log\log N_k}{\log(N_{k+1}/a_{N_{k+1}}\log\log N_{k+1})}$$

$$\leqslant c\left(1-\frac{N_k}{N_{k+1}}\right)(\log N_k)\left(\log\frac{N_{k+1}}{a_{N_{k+1}}\log\log N_{k+1}}\right)^{-1}\to 0.$$

因此

(2.3.43) $$\overline{\lim_{k\to\infty}}\ I_2(k)=0 \qquad \text{a.s.}$$

将（2.3.41）和（2.3.43）代入（2.3.40）得证（2.3.35）式，这也就完成了（2.3.30）式的证明．

其次来证

(2.3.44) $$\overline{\lim_{N\to\infty}}\max_{0\leqslant n\leqslant N-a_N}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n+k}-S_n|\leqslant 1 \qquad \text{a.s.}$$

设$r=r(\varepsilon)$是待定的正数，记

(2.3.45) $$C_N=[a_N/r],\quad n_i=iC_N.$$

写

(2.3.46) $$\max_{0\leqslant n\leqslant N-a_N}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n+k}-S_n|$$

$$\leqslant \max_{1\leqslant i\leqslant N/C_N}\max_{n_{i-1}\leqslant n<n_i}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}(|S_{n+k}-S_{n_i}|+|S_n-S_{n_i}|)$$

$$\leqslant 2\max_{1\leqslant i\leqslant N/C_N}\max_{n_{i-1}\leqslant n<n_i}\gamma_{nN}|S_n-S_{n_i}|$$

$$+\max_{1\leqslant i\leqslant N/C_N}\max_{n_{i-1}\leqslant n<n_i}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n_i+k}-S_{n_i}|.$$

于是

(2.3.47) $$P\{\max_{0\leqslant n\leqslant N-a_N}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n+k}-S_n|\geqslant 1+\varepsilon\}$$

$$\leqslant P\{\max_{1\leqslant i\leqslant N/C_N}\max_{n_{i-1}\leqslant n<n_i}\gamma_{nN}|S_n-S_{n_i}|\geqslant\varepsilon/4\}$$

$$+P\{\max_{1\leqslant i\leqslant N/C_N}\max_{n_{i-1}\leqslant n<n_i}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n_i+k}-S_{n_i}|$$

$$\geqslant 1+\varepsilon/2\}$$

$$\leqslant \frac{N}{C_N}\max_{1\leqslant i\leqslant N/C_N} P\{\max_{n_{i-1}\leqslant n<n_i}\gamma_{nN}|S_n-S_{n_i}|\geqslant \varepsilon/4\}$$

$$+\frac{N}{C_N}\max_{1\leqslant i\leqslant N/C_N} P\{\max_{n_{i-1}\leqslant n<n_i}\max_{1\leqslant k\leqslant s_N}\gamma_{nN}|S_{n_i+k}-S_{n_i}|$$

$$\geqslant 1+\varepsilon/2\}$$

$$=: J_1(N)+J_2(N).$$

先来估计$J_1(N)$。在定理的条件下，对于

$$g=1+\varepsilon/4,$$

存在$T>0$，使对任意的$k$，当$|t|\leqslant T$时

$$E\exp(tX_k)\leqslant \exp(g\sigma_k^2t^2/2).$$

于是若取引理2.2.1中的

$$x=\frac{\varepsilon}{4}\max_{n_{i-1}\leqslant n<n_i}\gamma_{nN}^{-1},\qquad G=g\sum_{j=n_{i-1}}^{n_i-1}\sigma_j^2,$$

利用条件（a)，当$N$充分大时有$0<x\leqslant GT$.因此由引理2.2.1

$$J_1(N)\leqslant \frac{N}{C_N}\max_{1\leqslant i\leqslant N/C_N}\exp\left\{-\frac{\varepsilon^2}{16}\left(\min_{n_{i-1}\leqslant n\leqslant n_i}\sigma_{nN}^2\right.\right.$$

$$\left.\left.\times\log(N/\sigma_{nN}^2\log\log N)\right)\Big/\left(g\sum_{j=n_{i-1}}^{n_i-1}\sigma_j^2\right)\right\}$$

$$\leqslant \frac{N}{C_N}\exp\left\{-c\varepsilon^2r\log(N/a_N\log\log N)\right\}$$

$$\leqslant \frac{rN}{a_N}\left(\frac{a_N\log\log N}{N}\right)^{c\varepsilon^2r}\leqslant r\,\frac{a_N}{N}(\log\log N)^2.$$

最后一个不等式可通过取足够大的$r=r(\varepsilon)$实现。再来估计$J_2(N)$.这时，当取$r$足够大时，对充分大的$N$

$$\left|\left(\min_{n_{i-1}\leqslant n<n_i}\sigma_{nN}^2\right)\Big/\left(\sum_{j=n_i+1}^{n_i+s_N}\sigma_j^2\right)-1\right|\leqslant \frac{\varepsilon}{4}.$$

类似于对$J_1(N)$的估计我们有

$$J_2(N)\leqslant\frac{N}{C_N}\max_{1\leqslant j\leqslant N/C_N}\exp\left\{-\left(1+\frac{\varepsilon}{2}\right)^2\left(\min_{n_{j-1}\leqslant n<n_j}\sigma_{nN}^2\right.\right.$$

$$\left.\left.\times\log(N/\sigma_{nN}^2\log\log N)\Big/\left(g\sum_{i=n_j+1}^{n_j+a_N}\sigma_i^2\right)\right)\right\}$$

$$=r\left(\frac{a_N}{N}\right)^{\varepsilon/4}(\log\log N)^{1+\varepsilon/4}.$$

将关于$J_1(N)$和$J_2(N)$的估计代入(2.3.47)式即得

$$P\{\max_{1\leqslant n\leqslant N-a_N}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n+k}-S_n|\geqslant1+\varepsilon\}$$

$$\leqslant2r\left(\frac{a_N}{N}\right)^{\varepsilon/4}(\log\log N)^2.$$

从假设(2.3.45)，存在正整数子列$\{N_j\}$使

$$\lim_{j\to\infty}P\{\max_{0\leqslant n\leqslant N_j-a_{N_j}}\max_{1\leqslant k\leqslant a_{N_j}}\gamma_{nN_j}|S_{n+k}-S_n|\geqslant1+\varepsilon\}=0.$$

由此即可推出(2.3.44)．

**注2.3.1** 当矩母函数不存在时，我们把条件(ii)改作

(ii)′存在非负不减函数$H(x)$，$x\geqslant0$，满足

对任意的$b>0$，$\sum_{n=1}^{\infty}P\{H(|X_n|)>bn\}<\infty$，

对每一$n$和任意的$\beta<1$，

$$E(H(|X_n|))^{\beta}\leqslant M<\infty,$$

对某个$\alpha>0$，$H(x)/x^{2+\alpha}$是增的，

对某个$A>0$，$\sum_{n=1}^{\infty}(\mathrm{inv}H(n))^{-A}<\infty$，

对任意的$\varepsilon>0$，$\varliminf_{x\to\infty}H(\varepsilon x)/H(x)>0$.

并把条件(a)改作

(a)′存在$a>0$使

$$a(\mathrm{inv}H(n))^2/\log n\leqslant a_n\leqslant n.$$

那么当$\{X_n\}$满足（i），（ii）′，$\{a_n\}$满足条件（a）′，（b）和（c）时定理2.3.3的结论仍成立。

## § 2.4 没有矩假设时的增量

在上两节中，关于随机变量序列的部分和的增量的所有极限结果都要求或者矩母函数存在，或者大于2阶的矩存在。但是强极限定理（原则上）只依赖于概率而不依赖矩（见Klass和Tomkins（1984））。一些人曾经研究过不加矩条件时的重对数律（例如，Klass和Teicher(1977)和Tomkins(1980)）。部分和的增量的a.s.极限性质可以看作是重对数律的推广和精细化。所以，当没有矩假设时，部分和的增量有多大是令人感兴趣的。林正炎(1990)首先讨论了这一问题，他的定理是具有矩条件时的结果的一个推广。林正炎和邵启满(1990)在很大程度上减弱了这一定理的条件。

设$\{X_n, n\geq 1\}$是独立但未必同分布的随机变量序列，$\{a_n, n\geq 1\}$是趋于无穷的不减正整数序列，记$S_n=\sum_{i=1}^{n} X_i$。又设$\{B_{nN}, n=0, 1, \cdots, N;\ N=1, 2, \cdots\}$是正数的三角组列，对固定的$n$，它关于$N$是不减的且当$N\to\infty$时关于$n$一致地趋于无穷。记$B_N=B_{0N}$，

$$b_N^2=2\{\log(B_{N+a_N}^2/B_{a_N}^2)+\log\log B_{a_N}^2\}.$$

对每一$N$和对应的$n$：$n+a_n\leq N<n+1+a_{n+1}$，定义

$B_N'=B_{n, n+a_n}$（故$B_{N+a_N}'=B_{N, N+a_N}$），$b_N'^2=2\{\log(B_{n+a_n}^2/B_N'^2)+\log\log B_N'^2\}$。对$\varepsilon>0$定义

$$X_{j\varepsilon}=(X_j\vee(-\varepsilon B_j' b_j'^{-1}))\wedge(\varepsilon B_j' b_j')$$

$$T_N(\varepsilon)=B_{N+a_N}^{-2}\sum_{j=1}^{N+a_N}\operatorname{Var}(X_{j\varepsilon}),$$

$$T_{nN}(\varepsilon)=B_{n, n+a_N}^{-2}\sum_{j=n+1}^{n+a_N}\operatorname{Var}(X_{j\varepsilon}),$$

$$T_-^{\varepsilon}=\lim_{\varepsilon\downarrow 0}\underline{\lim}_{N\to\infty} T_N(\varepsilon)\wedge T_{NN}(\varepsilon),$$

$$T_+^{\varepsilon}=\lim_{\varepsilon\downarrow 0}\overline{\lim_{N\to\infty}}\max_{0\leqslant n\leqslant N} T_{nN}(\varepsilon).$$

假设$T_+<\infty$。

**定理2.4.1** 假设下列条件被满足：对任给的$\varepsilon>0$，

(i) $\sum_{n=1}^{\infty} P\{|X_n|\geqslant \varepsilon B_n' b_n'\}<\infty$；

(ii) $\lim_{N\to\infty}\max_{0\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}(B_{a_N}b_N)^{-1}\left|\sum_{j=n+1}^{n+k} E\{X_j I(|X_j|\right.$

$\left.\leqslant \varepsilon B_j' b_j')\}\right|=0$；

(iii) $\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}}\max_{0\leqslant n\leqslant N}\sum_{j=n+1}^{n+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1}<|X_j|$

$<\varepsilon B_j' b_j')\ /(B_{a_N}^2((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-\beta})<\infty$

(某$\beta>0$)；

(iv) $\overline{\lim_{N\to\infty}}\ (\max_{0\leqslant n\leqslant N} B_{n,n+a_N})/(\min_{0\leqslant n\leqslant N} B_{n,n+a_N})<\infty$；

(v) 对某$A>0$和每一$N\geqslant 2$

$$B_{N+a_N}\leqslant AB_{N-1+a_{N-1}},\quad B_{a_N}\leqslant AB_{a_{N-1}}.$$

那么

$$T_-\leqslant\overline{\lim_{N\to\infty}}\frac{|S_{N+a_N}-S_N|}{B_{N,N+a_N}b_N}\leqslant\overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\frac{|S_{n+k}-S_n|}{B_{n,n+a_N}b_N}$$

$\leqslant T_+$ a.s.

**例** 设$\{X_n\}$是i.i.d.随机变量序列，有分布

$$P(X_1=-\sqrt{n})=P(X_1=\sqrt{n})=an^2/\log n,\qquad n=2,3,\cdots$$

其中 $a=\frac{1}{2}\left(\sum_{n=3}^{\infty}\frac{1}{n^2\log n}\right)^{-1}$。显然$EX_n^2=\infty$。取$a_n=n^2$，$B_{n,n+k}^2=2ak\log\log k$。于是$B_{n,n+a_N}^2=2aa_N\log\log a_N\sim 2aN^2\log\log N$。那么由定义我们有

$$B_{a_N}^2 \sim 2aN^2 \log\log N, \qquad b_N^2 \sim 2\log\log N,$$
$$B_N'^2 \sim 2aN \log\log N, \qquad b_N'^2 \sim 2\log\log N.$$

这样

$$\sum_{j=n+1}^{n+a_N} \operatorname{Var}(X_{j\varepsilon}) \sim \sum_{j=n+1}^{n+a_N} \left( \sum_{k=2}^{[\varepsilon^2 aj]} \frac{2a}{k\log k} + \sum_{k=[\varepsilon^2 aj]}^{\infty} \frac{2a^2\varepsilon^2 j}{k^2 \log k} \right)$$
$$\sim 2aa_N \log\log a_N.$$

因此 $T_-=T_+=1$. 不难验证条件 (i)—(v) 被满足.这里仅验证 (iii).对$0\leqslant n\leqslant N$, $k=a_N$,

$$\sum_{j=n+1}^{n+k} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| < \varepsilon B_j' b_j')$$

$$\leqslant \sum_{j=n+1}^{n+k} \sum_{i=[\varepsilon^2 aj]}^{[\varepsilon^2 aj(\log\log j)^2]} \frac{2a}{i\log i}$$

$$\leqslant 3a \sum_{j=n+1}^{n+k} \log\left(1+\frac{2\log(2\log\log j)}{\log(\varepsilon^2 aj)}\right)$$

$$\leqslant 7ak\frac{\log\log\log k}{\log k},$$

由此即知 (iii) 被满足(见邵启满1989).

为证明定理2.4.1, 我们需要下述引理(见邵启满, 1989).

**引理2.4.1** 设$\{\xi_n, n\geqslant 1\}$是独立随机变量序列,$E\xi_n=0$. 又设$\{a_n, n\geqslant 1\}$是趋于无穷的正数序列.假设存在正数的三角组列

$$\{\sigma_{nN}, n=0, 1, \cdots, N; N=1, 2, \cdots\},$$

对于固定的 $n$ 它关于$N$是不减的且当 $N\to\infty$ 时关于 $n$一致地趋于无穷. 令$\sigma_N^2=\min\limits_{0\leqslant n\leqslant N} \sigma_{na_N}^2$,

$$\beta_{nN}=\{2\sigma_{na_N}^2[\log(\sigma_{0,N+a_N}^2/\sigma_{na_N}^2)+\log\log\sigma_{na_N}^2]\}^{1/2}.$$

若下列条件被满足:

(a) 对某$A>0$, $\sigma_{0a_N}\leqslant A\sigma_{0a_{N-1}}$;

(b) $\varlimsup\limits_{N\to\infty} \max\limits_{1\leqslant n\leqslant N} \sum\limits_{j=n+1}^{n+a_N} E\xi_j^2/\sigma_{na_N}^2 \leqslant 1$;

(c) 存在$\varepsilon>0$使得

$$|\xi_N|\leqslant\varepsilon\{\sigma_N^2/(\log(\sigma_{0,N+a_N}^2/\sigma_N^2)+\log\log\sigma_N^2)\}^{1/2},$$

那么存在 $c(\varepsilon)\to0$ (当$\varepsilon\to0$) 使得

$$\overline{\lim_{N\to\infty}}\max_{0\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\left|\sum_{j=n+1}^{n+k}\xi_j\right|/\beta_{nN}\leqslant1+c(\varepsilon)\quad\text{a.s.}$$

定理2.4.1的证明

不失一般性，我们可假设$T_->0$。

首先，我们来给出若干下面要用到的事实。

对于$N$取$n$使$a_{n-1}<N+a_N\leqslant a_n$。由条件(V)知

$$\sum_{j=1}^{N+a_N}EX_j^2I(\varepsilon B_j'b_j'^{-1}<|X_j|\leqslant\varepsilon B_j'b_j')/B_{N+a_N}^2$$

$$\leqslant\sum_{j=1}^{a_n}EX_j^2I(\varepsilon B_j'b_j'^{-1}<|X_j|\leqslant\varepsilon B_j'b_j')/B_{a_{n-1}}^2$$

$$\leqslant A^2\sum_{j=1}^{a_n}EX_j^2I(\varepsilon B_j'b_j'^{-1}<|X_j|\leqslant\varepsilon B_j'b_j')/B_{a_n}^2.$$

类似地

$$\sum_{j=1}^{N+a_N}\mathrm{Var}(X_{j\varepsilon})/B_{N+a_N}^2\leqslant A^2\sum_{j=1}^{a_n}\mathrm{Var}(X_{j\varepsilon})/B_{a_n}^2.$$

因此由条件(iii)

$$(2.4.1)\quad\overline{\lim_{\varepsilon\downarrow0}}\,\overline{\lim_{N\to\infty}}\sum_{j=1}^{N+a_N}EX_j^2I(\varepsilon B_j'b_j'^{-1}<|X_j|\leqslant\varepsilon B_j'b_j')/B_{N+a_N}^2$$

$$\leqslant A^2\overline{\lim_{\varepsilon\downarrow0}}\,\overline{\lim_{N\to\infty}}\max_{0\leqslant n\leqslant N}\Big\{\sum_{j=n+1}^{n+a_N}EX_j^2I(\varepsilon B_j'b_j'^{-1}$$

$$<|X_j|\leqslant\varepsilon B_j'b_j')/[B_{a_N}^2((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-\beta}]\Big\}$$

$$\times((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-\beta}=0.$$

又有

$$(2.4.2)\qquad\overline{\lim_{\varepsilon\downarrow0}}\,\overline{\lim_{N\to\infty}}\sum_{j=1}^{N+a_N}\mathrm{Var}(X_{j\varepsilon})/B_{N+a_N}^2$$

$$\leqslant A^2\overline{\lim_{\varepsilon\downarrow0}}\,\overline{\lim_{N\to\infty}}\max_{0\leqslant n\leqslant N}T_{nN}(\varepsilon)=A^2T_+^2.$$

定义$N_j=\min\{n: B_{a_n}b_n\geqslant 2^j\}$，由此即得

$$B_{a_{N_j-1}}b_{N_j-1}<2^j\leqslant B_{a_{N_j}}b_{N_j}.$$

从条件（V），有常数$C>0$使对每一$N\geqslant 2$有

$$b_N\leqslant Cb_{N-1}.$$

所以

$$B_{a_{N_j}}b_{N_j}\leqslant ACB_{a_{N_j-1}}b_{N_j-1}$$

且

$$2^j\leqslant B_{a_{N_j}}b_{N_j}\leqslant AC2^j.$$

这样我们可得

$$B_{a_{N_{j+1}}}b_{N_{j+1}}/B_{a_{N_j}}b_{N_j}\leqslant 2AC. \tag{2.4.3}$$

进一步，我们或者有

$$B_{a_{N_j}}\geqslant 2^{j/2} \tag{2.4.4}$$

或者有

$$B_{a_{N_j}}<2^{j/2},\quad b_{N_j}\geqslant 2^{j/2}. \tag{2.4.5}$$

若(2.4.4)成立，那么

$$(B_{a_{N_j}}/B_{N_j+a_{N_j}})^{\varepsilon}/\log^{1+\varepsilon}B_{a_{N_j}}$$

$$\leqslant(\log B_{a_{N_j}})^{-(1+\varepsilon)}\leqslant\left(\frac{1}{2}j\log 2\right)^{-(1+\varepsilon)},$$

若(2.4.5)成立，那么

$$\log(B^2_{N_j+a_{N_j}}/B^2_{a_{N_j}})\geqslant 2^{j-1}-\log\log 2^j\geqslant 2^{j-2},$$

由此也有

$$(B_{a_{N_j}}/B_{N_j+a_{N_j}})^{\varepsilon}/(\log B_{a_{N_j}})^{1+\varepsilon}\leqslant j^{-(1+\varepsilon)}.$$

在任一情形下都有

$$(2.4.6)\qquad \sum_{j=1}^{\infty}(B_{a_{N_j}}/B_{N_j+a_{N_j}})^{\varepsilon}/\log^{1+\varepsilon}B_{a_{N_j}}<\infty.$$

此外，因为从条件（V）和(2.4.2)，当$N$充分大时

$$\sum_{j=1}^{N+a_N}\mathrm{Var}(X_{j\varepsilon})\Big/\sum_{j=1}^{N-1+a_{N-1}}\mathrm{Var}(X_{j\varepsilon})$$

$$=(B^2_{N+a_N}/B^2_{N-1+a_{N-1}})\Big(\sum_{j=1}^{N+a_N}\mathrm{Var}(X_{j\varepsilon})/B^2_{N+a_N}\Big)$$

$$\Big/\Big(\sum_{j=1}^{N-1+a_{N-1}}\mathrm{Var}(X_{j\varepsilon})/B^2_{N-1+a_{N-1}}\Big)$$

$$\leqslant A^2\cdot 2A^2T_+^2/(T_-^2/2)=4A^4T_+^2/T_-^2.$$

所以存在$\delta>0$和$Q>0$使得关于$0<\varepsilon<\delta$一致地有

$$(2.4.7)\qquad \varlimsup_{N\to\infty}\sum_{j=1}^{N+a_N}\mathrm{Var}(X_{j\varepsilon})\Big/\sum_{j=1}^{N-1+a_{N-1}}\mathrm{Var}(X_{j\varepsilon})\leqslant Q.$$

利用这些事实，我们来证明定理的结论.

对于给定的$\delta>0$，记$\varepsilon=\varepsilon(\delta)$，它在下面确定.定义

$c_n=\varepsilon B_n'b_n'^{-1}$，$d_n=\varepsilon B_n'b_n'$，$X_n'=X_{n\varepsilon}$，

$Y_n=(X_n-c_n\mathrm{sign}X_n)I(c_n<|X_n|\leqslant d_n)$，$Z_n=X_n'+Y_n$，

$$S_n'=\sum_{k=1}^{n}(X_k'-EX_k'),\quad U_n=\sum_{k=1}^{n}(Y_k-EY_k),$$

$$V_n=\sum_{k=1}^{n}(Z_k-EZ_k).$$

那么

$$Z_n=X_nI(|X_n|\leqslant d_n)+c_n\mathrm{sign}X_nI(|X_n|>d_n),$$

$$|X_n-Z_n|\leqslant|X_n|I(|X_n|>d_n).$$

故由条件（i）有

$$(2.4.8)\qquad P\{X_n\neq Z_n,\ \text{i.o.}\}=0.$$

因此代替$X_n$，我们仅需考察$Z_n$.从条件（i），（ii）和$c_n$的定义，我们有

(2.4.9) $\lim_{N\to\infty}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\frac{1}{B_{n,n+a_N}b_N}\left|\sum_{j=n+1}^{n+k}EZ_j\right|=0.$

结合（2.4.8）和（2.4.9）可知定理的结论等价于

(2.4.10) $$T_-\leqslant\overline{\lim_{N\to\infty}}\frac{|V_{N+a_N}-V_N|}{B_{N,N+a_N}b_N}$$

$$\leqslant\overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\frac{|V_{n+k}-V_n|}{B_{n,n+a_N}b_N}\leqslant T_+\quad \text{a.s.}$$

第一步，我们证明

(2.4.11) $$\overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\frac{|U_{n+k}-U_n|}{B_{n,n+a_N}b_N}\leqslant\delta\quad \text{a.s.}$$

设$r=r(\delta)$是一个正整数，它在下面给定。令 $D_N=B^2_{a_N}((B^2_{N+a_N}/B^2_{a_N})\log B^2_{a_N})^{-\beta}$，$Y_0=0$。定义$n$，$N$和$r$的函数如下：

$$n_r=\max\Big\{k:\sum_{j=1}^{k}\mathrm{Var}Y_j\leqslant iD_N/r,$$

其中 $i$ 满足$\frac{i}{r}D_N\leqslant\sum_{j=1}^{n}\mathrm{Var}\,Y_j\leqslant\frac{i+1}{r}D_N\Big\}$.

令$U_0=0$。写

(2.4.12) $$|U_{n+k}-U_n|$$

$$\leqslant|U_{n+k}-U_{(n+k)_r}|+|U_{(n+k)_r}-U_{n_r}|+|U_n-U_{n_r}|.$$

考察上式右边的第一项，由条件（iv），存在常数$H>0$使对每一$N$成立

(2.4.13) $$(\max_{0\leqslant n\leqslant N}B_{n,n+a_N})/(\min_{0\leqslant n\leqslant N}B_{n,n+a_N})\leqslant H.$$

那么我们有

(2.4.14) $$P\Big\{\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\frac{1}{B_{n,n+a_N}b_N}|U_{n+k}-U_{(n+k)_r}|\geqslant\frac{\delta}{2ACH}\Big\}$$

$$\leqslant\Big\{r\sum_{j=1}^{N+a_N}\mathrm{Var}Y_j/D_N+1\Big\}\max_{1\leqslant n\leqslant N+a_N}P\Big\{|U_n-U_{n_r}|$$

$$\geqslant\frac{\delta}{2ACH^2}B_{a_N}b_N\Big\}.$$

从条件（iii）对某一$M>0$

(2.4.15)
$$\sum_{j=1}^{N+a_N}\mathrm{Var}Y_j$$
$$\leqslant 2\sum_{j=1}^{N+a_N}\{EX_j^2I(c_j<|X_j|\leqslant d_j)$$
$$+c_j^2P(c_j<|X_j|\leqslant d_j)\}$$
$$\leqslant 2MB_{N+a_N}^2.$$

估计(2.4.14)的右边中的概率．设
$$\varepsilon=\beta\delta/(16ACH^3),\quad t=\beta b_N/(4\varepsilon H)=4ACH^2b_N/\delta.$$
那么对$j\leqslant N+a_N$，我们有$d_j\leqslant\varepsilon HB_{a_N}b_N$和
$$E\exp\{t(Y_j-EY_j)/B_{a_N}\}$$
$$\leqslant 1+\frac{t^2}{2B_{a_N}^2}\mathrm{Var}Y_j\left\{1+\frac{1}{3}\left(\frac{2td_j}{B_{a_N}}\right)+\frac{1}{12}\left(\frac{2td_j}{B_{a_N}}\right)^2+\cdots\right\}$$
$$\leqslant 1+\frac{t^2}{2B_{a_N}^2}\mathrm{Var}Y_j\cdot\exp\left\{\frac{\beta}{4}b_N^2\right\}$$
$$\leqslant\exp\left\{\frac{8A^2C^2H^4}{\delta^2B_{a_N}^2}b_N^2\left(\frac{B_{N+a_N}^2}{B_{a_N}^2}\log B_{a_N}^2\right)^{\beta/2}\mathrm{Var}Y_j\right\}.$$

令 $l_n=\max\{m_t m_r=n_r\}$．利用Lévy极大不等式我们得对每一充分大的$N$有

(2.4.16)
$$P\left\{\max_{m_r=n_r}|U_m-U_{n_r}|\geqslant\frac{\delta}{2ACH^2}B_{a_N}b_N\right\}$$
$$\leqslant 2P\left\{|U_{l_n}-U_{n_r}|\geqslant\frac{\delta}{2ACH^2}B_{a_N}b_N-2\sqrt{\mathrm{Var}(U_{l_n}-U_{n_r})}\right\}$$
$$\leqslant 2P\left\{|U_{l_n}-U_{n_r}|\geqslant\frac{\delta}{3ACH^2}B_{a_N}b_N\right\}$$
$$\leqslant 2\exp\left(-\frac{\delta tb_N}{3ACH^2}\right)\prod_{j=n_r+1}^{l_n}E\exp\left\{\frac{t}{B_{a_N}}(Y_j-EY_j)\right\}$$

$$\leqslant 2\exp\{-4b_N^2/3+o(b_N^2)\}$$

$$\leqslant ((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-2},$$

这里最后第二个不等号利用了 $n_r$ 的定义和条件（iv）. 结合（2.4.14），（2.4.15）和（2.4.16）得

$$(2.4.17)\quad P\left\{\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\frac{1}{B_{n,n+a_N}b_N}|U_{n+k}-U_{(n+k)_r}|\geqslant\frac{\delta}{2ACH}\right\}$$

$$\leqslant 5rM(B_{a_N}^2/B_{N+a_N}^2)^{1-\beta}/(\log B_{a_N}^2)^{2-\beta}.$$

由（2.4.6）我们有

$$\sum_{j=1}^{\infty}P\left\{\max_{1\leqslant n\leqslant N_j}\max_{1\leqslant k\leqslant a_{N_j}}\frac{1}{B_{n,n+a_{N_j}}b_{N_j}}|U_{n+k}-U_{(n+k)_r}|\right.$$

$$\left.\geqslant\frac{\delta}{2ACH}\right\}<\infty.$$

由此即得

$$(2.4.18)\quad\overline{\lim_{j\to\infty}}\max_{1\leqslant n\leqslant N_j}\max_{1\leqslant k\leqslant a_{N_j}}\frac{1}{B_{n,n+a_{N_j}}b_{N_j}}|U_{n+k}-U_{(n+k)_r}|$$

$$\leqslant\frac{\delta}{2ACH}\qquad\text{a.s.}$$

进一步，注意到(2.4.18)中两个max的变化范围随着 $j$ 的增大而增大的，利用（2.4.3）和（2.4.13），由（2.4.18）可得

$$(2.4.19)\quad\overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\frac{1}{B_{n,n+a_N}b_N}|U_{n+k}-U_{(n+k)_r}|\leqslant\delta\quad\text{a.s.}$$

（2.4.12）式右边的第二和第三项可同样处理（除去对第二项需应用条件（iii)外)，我们有类似的不等式.（2.4.11）被证明. 这样不等式（2.4.10）等价于对任一$0<\delta<1/2$

$$(2.4.20)\quad(1-2\delta)T_-\leqslant\overline{\lim_{N\to\infty}}\frac{|S'_{N+a_N}-S'_N|}{B_{N,N+a_N}b_N}$$

$$\leqslant\overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\frac{|S'_{n+k}-S'_n|}{B_{n,n+a_N}b_N}\leqslant(1+\delta)T_+\quad\text{a.s.}$$

容易验证，$\{X_n'\}$满足引理2.4.1的条件. 由此从$T_+$的定义我们得（2.4.20）的右边不等式.

其次，我们来证（2.4.20）的左边不等式.令

$$v_{n,n+a_N}^2=\sum_{j=n+1}^{n+a_N}\mathrm{Var}X_j',\quad v_{a_N}^2=\max_{1\leqslant n\leqslant N} v_{n,n+a_N}^2.$$

对于$j\leqslant N+a_N$，我们有

(2.4.21) $|X_j'-EX_j'|\leqslant 2c_j\leqslant 2c_{N+a_N}$

$$=2\varepsilon b_{N+a_N}'^{-1}(B_{N,N+a_N}/v_{N,N+a_N})v_{N,N+a_N}.$$

注意到 $0<T_-<\infty$，当 $N\to\infty$时我们有 $B_{N,N+a_N}b_N/v_{N,N+a_N}\to\infty$ 且对某常数$Q$和每一$N\geqslant 1$

$$(\varepsilon b_{N+a_N}'^{-1}B_{N,N+a_N}v_{N,N+a_N}^{-1})(B_{N,N+a_N}b_Nv_{N,N+a_N}^{-1})\leqslant Q\varepsilon.$$

因此，若$\varepsilon=\varepsilon(\delta)$ 充分小，我们利用引理2.2.1(b) 得

(2.4.22) $P\left\{\dfrac{1}{B_{N,N+a_N}b_N}\ |S_{N+a_N}'-S_N'|\geqslant(1-\delta)T_-\right\}$

$$\geqslant\exp\left\{-\frac{(1+\delta)(1-\delta)^2T_-^2B_{N,N+a_N}^2b_N^2}{2v_{N,N+a_N}^2}\right\}$$

$$\geqslant\left(\frac{B_{a_N}^2}{B_{N+a_N}^2\log B_{a_N}^2}\right)^{1-\delta}\geqslant\frac{B_{a_N}^2}{B_{N+a_N}^2\log B_{N+a_N}^2}.$$

按$T_-$和$T_+$的定义，我们可选$\varepsilon$使（2.4.22）式的右边大于

(2.4.23) $\qquad Rv_{N,N+a_N}^2/(v_{0,N+a_N}^2\log v_{0,N+a_N}^2)$,

其中$R$是一个大于 0 的常数. 设 $0<\eta<\delta\wedge(\delta^2T_-^2/4T_+^2)$, $N_1=1$, 由下式定义$N_{k+1}$:

(2.4.24) $\qquad N_{k+1}=\min\{n: v_{0,n}^2+\eta v_{n,n+a_n}^2\geqslant v_{0,N_k+a_{N_k}}^2\}$.

那么我们有

(2.4.25) 对每一$k$ $\quad v_{0,N_{k+1}}^2+\eta v_{N_{k+1},N_{k+1}+a_{N_{k+1}}}^2\geqslant v_{0,N_k+a_{N_k}}^2$,

(2.4.26) 对每一$n<N_{k+1}$ $\quad v_{0,n}^2+\eta v_{n,n+a_n}^2<v_{0,N_k+a_{N_k}}^2$.

因此，对每一$k\geqslant 1$我们有$N_{k+1}>N_k$和$N_{k+1}+a_{N_{k+1}}>N_k+a_{N_k}$.

我们先证

$$(2.4.27)\quad \sum_{k=1}^{\infty} v_{N_k, N_k+a_{N_k}}^2 / (v_{0, N_k+a_{N_k}}^2 \log v_{0, N_k+a_{N_k}}^2) = \infty.$$

由（2.4.26），当$k$充分大时我们得

$$(2.4.28)\quad v_{0,N_{k-1}+a_{N_{k-1}}}^2 \geqslant v_{0,N_{k-1}}^2 = v_{0,N_k}^2 - \operatorname{Var} X_k'$$

$$\geqslant v_{0,N_k}^2 - \varepsilon^2 B_{N_k}'^2 b_{N_k}'^{-2} \geqslant v_{0,N_k}^2 - H\varepsilon^2 B_{N_k, N_k+a_{N_k}}^2 b_{N_k}'^{-2}$$

$$\geqslant v_{0,N_k}^2 - v_{N_k, N_k+a_{N_k}}^2 = v_{0,N_k+a_{N_k}}^2 - 2v_{N_k, N_k+a_{N_k}}^2,$$

又从（2.4.26）和（2.4.7）有

$$(2.4.29)\quad v_{0,N_{k-1}+a_{N_{k-1}}}^2 \geqslant \eta v_{0,N_{k-1}+a_{N_{k-1}}}^2 \geqslant \frac{\eta}{C} v_{0,N_k+a_{N_k}}^2.$$

现在利用（2.4.28）和（2.4.29），我们有

$$\sum_{k=2}^{\infty} v_{N_k, N_k+a_{N_k}}^2 / (v_{0,N_k+a_{N_k}}^2 \log v_{0,N_k+a_{N_k}}^2)$$

$$\geqslant \frac{1}{2} \sum_{k=2}^{\infty} (v_{0,N_k+a_{N_k}}^2 - v_{0,N_{k-1}+a_{N_{k-1}}}^2)$$

$$/(v_{0,N_k+a_{N_k}}^2 \log v_{0,N_k+a_{N_k}}^2)$$

$$\geqslant \frac{\eta}{2C} \sum_{k=2}^{\infty} (v_{0,N_k+a_{N_k}}^2 - v_{0,N_{k-1}+a_{N_{k-1}}}^2)$$

$$\Big/ \left( v_{0,N_{k-1}+a_{N_{k-1}}}^2 \log\left( \frac{C}{\eta} v_{0,N_{k-1}+a_{N_{k-1}}}^2 \right) \right)$$

$$\geqslant \frac{\eta}{2C} \sum_{k=2}^{\infty} \int_{\sigma_{0,N_{k-1}+a_{N_{k-1}}}^2}^{\sigma_{0,N_k+a_{N_k}}^2} \frac{1}{x \log x} dx = \infty,$$

这就证明了（2.4.27）式成立.

令$G=\{k: N_k \geqslant N_{k-1}+a_{N_{k-1}}\}$，$K=\{k: N_k < N_{k-1}+a_{N_{k-1}}\}$.

为证明（2.4.20），我们分两种情形讨论如下.

**情形1** 假设

(2.4.30) $\sum_{k\in G} v^2_{N_k,N_k+a_{N_k}}/(v^2_{0,N_k+a_{N_k}}\log v^2_{0,N_k+a_{N_k}})=\infty.$

那么，由（2.4.23）

(2.4.31) $\sum_{k\in G} P\left\{\frac{1}{B_{N_k,N_k+a_{N_k}}b_{N_k}}|S'_{N_k+a_{N_k}}-S'_{N_k}|\geqslant(1-\delta)T_-\right\}$

$=\infty.$

注意到 $\{S'_{N_k+a_{N_k}}-S'_{N_k}, k\in G\}$ 是独立随机变量序列，由(2.4.31)即得

$$\varlimsup_{\substack{k\to\infty\\k\in G}}\frac{|S'_{N_k+a_{N_k}}-S'_{N_k}|}{B_{N_k,N_k+a_{N_k}}b_{N_k}}\geqslant(1-\delta)T_-\qquad \text{a.s.}$$

因此就有

$$\varlimsup_{N\to\infty}\frac{|S'_{N+a_N}-S'_N|}{B_{N,N+a_N}b_N}\geqslant(1-\delta)T_-\qquad \text{a.s.}$$

即得（2.4.20）的左边成立.

**情形2　假设**

(2.4.32) $\sum_{k\in G} v^2_{N_k,N_k+a_{N_k}}/(v^2_{0,N_k+a_{N_k}}\log v^2_{0,N_k+a_{N_k}})<\infty.$

那么由（2.4.27）有

(2.4.33) $\sum_{k\in K} v^2_{N_k,N_k+a_{N_k}}/(v^2_{0,N_k+a_{N_k}}\log v^2_{0,N_k+a_{N_k}})=\infty.$

从（2.4.25）知对每一$k\in K$

(2.4.34) $\quad 0\leqslant v^2_{0,N_{k-1}+a_{N_{k-1}}}-v^2_{0,N_k}\leqslant\eta v^2_{N_k,N_k+a_{N_k}}.$

因此

(2.4.35) $(1-\eta)v^2_{N_k,N_k+a_{N_k}}\leqslant v^2_{0,N_k+a_{N_k}}-v^2_{0,N_{k-1}+a_{N_{k-1}}}$

$$\leqslant v^2_{N_k,N_k+a_{N_k}}.$$

写

$$|S'_{N_k+a_{N_k}}-S'_{N_k}|$$

$$\geqslant|S'_{N_k+a_{N_k}}-S'_{N_{k-1}+a_{N_{k-1}}}|-|S'_{N_{k-1}+a_{N_{k-1}}}-S'_{N_k}|.$$

考察上式右边第一项.注意到（2.4.35），如在(2.4.22)中一样，应用引理2.2.1(b) 我们得

$$\sum_{k\in K}P\left\{\frac{1}{B_{N_k,N_k+aN_k}b_{N_k}}|S'_{N_k+aN_k}-S'_{N_{k-1}+aN_{k-1}}|\right.$$

$$\left.\geqslant(1-\delta)T_-\right\}$$

$$\geqslant\sum_{k\in K}(B^2_{aN_k}/(B^2_{N_k+aN_k}\log B^2_{N_k+aN_k}))^{(1-\delta)/(1-\eta)}$$

$$\geqslant c\sum_{k\in K}v^2_{N_k,N_k+aN_k}/(v^2_{0,N_k+aN_k}\log v^2_{0,N_k+aN_k})=\infty,$$

由此即得

(2.4.36) $$\overline{\lim_{\substack{k\to\infty\\k\in K}}}\frac{|S'_{N_k+aN_k}-S'_{N_{k-1}+aN_{k-1}}|}{B_{N_k,N_k+aN_k}b_{N_k}}\geqslant(1-\delta)T_-\qquad\text{a.s.}$$

利用（2.4.34）并注意到 $\eta<\delta^2T^2_-/4T^2_+$，对充分大的 $k$ 和充分小的$e$有

$$\prod_{j=N_k+1}^{N_{k-1}+aN_{k-1}}E\exp\left\{\frac{3b_{N_k}}{\delta T_-B_{N_k,N_k+aN_k}}(X'_j-EX'_j)\right\}$$

$$\leqslant\exp\left\{\frac{3b^2_{N_k}}{\delta^2T^2_-B^2_{N_k+aN_k}}\sum_{j=N_k+1}^{N_{k-1}+aN_{k-1}}\operatorname{Var}X'_j\right\}$$

$$\leqslant\exp\{3\eta b^2_{N_k}v^2_{N_k,N_k+aN_k}/\delta^2T^2_-B^2_{N_k,N_k+aN_k}\}$$

$$\leqslant\exp\{4\eta T^2_+b^2_{N_k}/\delta^2T^2_-\}$$

$$\leqslant\exp(b^2_{N_k}).$$

因此，利用$T_+$和$T_-$的定义，对充分大的$k$和充分小的$e$，我们有

(2.4.37) $$P\left\{\frac{1}{B_{N_k,N_k+aN_k}b_{N_k}}|S'_{N_{k-1}+aN_{k-1}}-S'_{N_k}|\geqslant\delta T_-\right\}$$

$$\leqslant\exp\{-3b^2_{N_k}+b^2_{N_k}\}=B^4_{aN_k}/(B^4_{N_k+aN_k}\log^2B^2_{aN_k})$$

$$\leqslant v^2_{N_k,N_k+aN_k}/(v^2_{0,N_k+aN_k}\log^2v^2_{N_k,N_k+aN_k})$$

$$\leqslant v_{N_k^2,N_k+aN_k}/(v^2_{0,N_k+aN_k}\log^2v^2_{0,N_k+aN_k}).$$

由 (2.4.25)

$$v^2_{N_k,N_k+a_{N_k}}=v^2_{0,N_k+a_{N_k}}-v^2_{0,N_k}$$

$$\leqslant v^2_{0,N_k+a_{N_k}}-v^2_{0,N_{k-1}+a_{N_{k-1}}}+\eta v^2_{N_k,N_k+a_{N_k}}.$$

故若$\eta<1/2$，就有

$$v^2_{N_k,N_k+a_{N_k}}\leqslant 2(v^2_{0,N_k+a_{N_k}}-v^2_{0,N_{k-1}+a_{N_{k-1}}}).$$

利用此不等式，仿照 (2.4.27) 的证明，我们可得

$$\sum_{k=1}^{\infty}v^2_{N_k,N_k+a_{N_k}}/(v^2_{0,N_k+a_{N_k}}\log^2 v^2_{0,N_k+a_{N_k}})<\infty.$$

所以

$$\sum_{k\in K}v^2_{N_k,N_k+a_{N_k}}/(v^2_{0,N_k+a_{N_k}}\log^2 v^2_{0,N_k+a_{N_k}})<\infty.$$

这样，从 (2.4.37) 我们有

$$(2.4.38)\quad \varlimsup_{\substack{k\to\infty\\ k\in K}}\frac{|S'_{N_{k-1}+a_{N_{k-1}}}-S'_{N_k}|}{B_{N_k,N_k+a_{N_k}}b_{N_k}}\leqslant\delta T_-\quad \text{a.s.}$$

综合 (2.4.36) 和 (2.4.38) 我们得

$$\varlimsup_{\substack{k\to\infty\\ k\in K}}\frac{|S'_{N_k+a_{N_k}}-S'_{N_k}|}{B_{N_k,N_k+a_{N_k}}b_{N_k}}\geqslant(1-2\delta)T_-\quad \text{a.s.}$$

因此

$$\varlimsup_{N\to\infty}\frac{|S'_{N+a_N}-S'_N|}{B_{N,N+a_N}b_N}\geqslant(1-2\delta)T_-\quad \text{a.s.}$$

即得 (2.4.20) 式的左边不等式成立.定理证毕.

## § 2.5　独立随机变量部分和的增量有多小?

设$\{X_n, n\geqslant 1\}$是i.i.d.随机变量序列,$EX_1=0$且$EX_1^2=1$.钟开莱(1948)首先讨论了 $\max\limits_{1\leqslant i\leqslant n}|S_i|$ 的下极限，证明着

$$(2.5.1)\qquad \varliminf_{n\to\infty}\left\{\frac{8\log\log n}{\pi^2 n}\right\}^{1/2}\max_{1\leqslant i\leqslant n}|S_i|=1\quad \text{a.s.}$$

Csáki(1978)指出(1.5.1) 的逆也正确.Csörgő和Révész (1981)考察了部分和的增量并给出了定理2.1.4.我们在§2.1中已指出他们所给的这一定理的证明是行不通的.在本节中,我们不仅给出了它的严格证明,且在较弱的假设下推广到了不同分布情形.

**定理2.5.1**(邵启满,1990) 假设$\{X_n, n\geqslant 1\}$是独立随机变量序列,对每一$n$,$EX_n=0$,$EX_n^2\geqslant C_0>0$,而且$\{X_n^2, n\geqslant 1\}$一致可积.又设正整数序列$\{a_n\}$满足下列条件:

(i) $1\leqslant a_N\leqslant N$;

(ii) 当$N\to\infty$时,$a_N/\log N\to\infty$.

记 $\sigma_{nk}^2=\sum\limits_{i=n+1}^{n+k} EX_i^2$,$S_0=0$.那么我们有

$$\lim_{N\to\infty}\min_{0\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n+k}-S_n|=1 \quad \text{a.s.} \tag{2.5.2}$$

$$\lim_{N\to\infty}\max_{1\leqslant k\leqslant a_N}\gamma_{NN}|S_{N+k}-S_N|=1 \tag{2.5.3}$$

其中$\gamma_{nN}=\{8(\log N/a_N+\log\log N)/\pi^2\sigma_{na_N}^2\}^{1/2}$.

若还满足

(iii) $\lim\limits_{N\to\infty}(\log N/a_N)/\log\log N=\infty$,

那么

$$\lim_{N\to\infty}\min_{0\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n+k}-S_n|=1 \quad \text{a.s.} \tag{2.5.4}$$

为证明本定理,我们需要下列引理.

**引理2.5.1** 设$\{X_n\}$是独立随机变量序列.假如存在$\varepsilon>0$,$0<\alpha<1$和正整数$p\geqslant 1$使得对某$x>0$

$$P\{\max_{1\leqslant k\leqslant p}|S_k|\geqslant\varepsilon x\}\leqslant\alpha. \tag{2.5.5}$$

那么

$$P\left\{\bigcup_{n=0}^{p}(\max_{1\leqslant k\leqslant N}|S_{n+k}-S_n|\leqslant x)\right\} \tag{2.5.6}$$

$$\leqslant\frac{1}{1-\alpha}P\{\max_{1\leqslant k\leqslant N}|S_k|\leqslant(1+\varepsilon)x\}.$$

**证** 记

$$E_p=\{\max_{1\leqslant k\leqslant N}|S_{p+k}-S_p|\leqslant x\},$$

$$E_i=\bigcap_{i<n\leqslant p}\{\max_{1\leqslant k\leqslant N}|S_{n+k}-S_n|>x\}$$

$$\cap\{\max_{1\leqslant k\leqslant N}|S_{i+k}-S_i|\leqslant x\},\quad i=0,1,\cdots,p-1.$$

显然

$$\bigcup_{n=0}^{p}\{\max_{1\leqslant k\leqslant N}|S_{n+k}-S_n|\leqslant x\}=\bigcup_{n=0}^{p}E_n$$

$$\subset\{\max_{1\leqslant k\leqslant N}|S_k|<(1+\varepsilon)x\}\cup\left(\bigcup_{n=1}^{p}(E_n\cap\{\max_{1\leqslant k\leqslant N}|S_k|\right.$$

$$\left.\geqslant(1+\varepsilon)x\})\right)$$

$$\subset\{\max_{1\leqslant k\leqslant N}|S_k|<(1+\varepsilon)x\}\cup\left(\bigcup_{n=1}^{p}(E_n\cap\{\max_{1\leqslant k\leqslant n}|S_k|\right.$$

$$\left.\geqslant(1+\varepsilon)x\})\right)$$

$$\cup\left(\bigcup_{n=1}^{p}(E_n\cap\{\max_{n<k\leqslant N}|S_k|\geqslant(1+\varepsilon)x\})\right)$$

$$\subset\{\max_{1\leqslant k\leqslant N}|S_k|<(1+\varepsilon)x\}\cup\left(\bigcup_{n=1}^{p}(E_n\cap\{\max_{1\leqslant k\leqslant n}|S_k|\right.$$

$$\left.\geqslant(1+\varepsilon)x\})\right)\cup\left(\bigcup_{n=1}^{p}(E_n\cap\{|S_n|\geqslant\varepsilon x\})\right)$$

$$\subset\{\max_{1\leqslant k\leqslant N}|S_k|<(1+\varepsilon)x\}\cup\left(\bigcup_{n=1}^{p}(E_n\cap\{\max_{1\leqslant k\leqslant n}|S_k|\geqslant\varepsilon x\})\right).$$

注意到$E_n$和$\{\max\limits_{1\leqslant k\leqslant n}|S_k|\geqslant\varepsilon x\}$是独立的，我们有

$$P\left\{\bigcup_{n=0}^{p}(\max_{1\leqslant k\leqslant N}|S_{n+k}-S_n|\leqslant x)\right\}$$

$$\leqslant P\{\max_{1\leqslant k\leqslant N}|S_k|\leqslant(1+\varepsilon)x\}+\sum_{n=1}^{p}P(E_n)P\{\max_{1\leqslant k\leqslant n}|S_k|\geqslant\varepsilon x\}$$

$$\leqslant P\{\max_{1\leqslant k\leqslant N}|S_k|\leqslant(1+\varepsilon)x\}+a\sum_{n=1}^{p}P(E_n)$$

$$\leqslant P\{\max_{1\leqslant k\leqslant N}|S_k|\leqslant(1+\varepsilon)x\}+aP\left\{\bigcup_{n=0}^{p}(\max_{1\leqslant k\leqslant N}|S_{n+k}-S_n|\leqslant x)\right\}.$$

得证引理成立.

**引理2.5.2** 设$\{X_n\}$是满足定理2.5.1的条件的随机变量序列. 假如 $\max\limits_{1\leqslant n\leqslant N}x_{nN}\to 0$ 且 $\min\limits_{1\leqslant n\leqslant N}x_{nN}\sigma_{na_N}\to\infty$ $(N\to\infty)$,

那么关于$n$, $1\leqslant n\leqslant N$, 一致地有

$$\log P\{\max_{1\leqslant k\leqslant a_N}|S_{n+k}-S_n|\leqslant x_{nN}\sigma_{na_N}\}\sim-\pi^2/8x^2_{nN}.$$

引理2.5.2的证明可在邵启满(1990)中找到.

定理2.5.1的证明分成三步给出.

**第一步** 对任给$0<\varepsilon<1/4$, 我们有

$$\varliminf_{N\to\infty}\min_{0\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n+k}-S_n|\geqslant 1-3\varepsilon\qquad\text{a.s.}\tag{2.5.7}$$

**证** 从$\{X_n^2\}$的一致可积性, 存在常数$C_2>0$使

$$C_0\leqslant EX_n^2\leqslant C_2\tag{2.5.8}$$

对每一$n\geqslant1$成立. 设

$$1<\theta<1+\varepsilon C_0/4C_2.\tag{2.5.9}$$

定义

$$H_k=\{N:[\theta^k]<a_N\leqslant[\theta^{k+1}]\},$$
$$M_k=\max\{N:N\in H_k\},$$
$$D_L=[\log_\theta(\inf_{N\geqslant L}a_N)]-1.$$

注意到

(2.5.10) $\inf\limits_{N\geqslant L}\min\limits_{0\leqslant n\leqslant N}\max\limits_{1\leqslant i\leqslant a_N}\gamma_{nN}|S_{n+i}-S_n|$

$$\geqslant \inf_{k\geqslant D_L} \min_{N\in H_k} \min_{0\leqslant n\leqslant N} \max_{1\leqslant i\leqslant a_N} \gamma_{nN}|S_{n+i}-S_n|$$

$$\geqslant \inf_{k\geqslant D_L} \min_{0\leqslant n\leqslant M_k} \max_{1\leqslant i\leqslant \theta^k} \gamma'_{nk}|S_{n+i}-S_n|,$$

其中$\gamma'_{nk}=\{8(\log(n\vee\theta^{k+1})/\theta^{k+1}+\log\log\theta^{k+1})/\pi^2\sigma^2_{n,[\theta^{k+1}]}\}^{1/2}$。

设$p_0=0$, $p_j=[C_0\varepsilon^3\theta^k/32C_2\log(jk)]$, $j=1,\cdots,m_k:=\min\left\{n:\right.$

$\left.\sum_{j=0}^{n}p_j\geqslant M_k\right\}$.容易看出

$$q_j:=\sum_{i=0}^{j}p_i\sim jC_0\varepsilon^3\theta^k/32C_2\log(jk). \tag{2.5.11}$$

从 (2.5.8) 我们有

$$\lim_{k\to\infty}\max_{0\leqslant j<m_k}\max_{q_j\leqslant n\leqslant q_{j+1}}\gamma'_{q_jk}/\gamma'_{nk}=1$$

由此及Lévy极大不等式，对充分大的$k$有

$$P\{\max_{0\leqslant i\leqslant p_{j+1}}|S_{q_j+i}-S_{q_j}|\geqslant\varepsilon\min_{q_j\leqslant n\leqslant q_{j+1}}1/\gamma'_{nk}\}$$

$$\leqslant P\{\max_{0\leqslant i\leqslant p_{j+1}}|S_{q_j+i}-S_{q_j}|\geqslant\varepsilon/2\gamma'_{q_jk}\}$$

$$\leqslant 2P\{|S_{q_j+p_{j+1}}-S_{q_j}|\geqslant\varepsilon/4\gamma'_{q_jk}\}$$

$$\leqslant 32C_2p_{j+1}\gamma'^2_{q_jk}/\varepsilon^2\leqslant 1/2.$$

因此，应用引理2.5.1，我们有

$$P\{\min_{0\leqslant n\leqslant M_k}\max_{1\leqslant i\leqslant\theta^k}\gamma'_{nk}|S_{n+i}-S_n|\leqslant 1-3\varepsilon\} \tag{2.5.12}$$

$$\leqslant\sum_{j=0}^{m_k-1}P\{\min_{q_j\leqslant n\leqslant q_{j+1}}\max_{1\leqslant i\leqslant\theta^k}\gamma'_{nk}|S_{n+i}-S_n|\leqslant 1-3\varepsilon\}$$

$$\leqslant 2\sum_{j=0}^{m_k-1}P\{\max_{1\leqslant i\leqslant\theta^k}\gamma'_{q_jk}|S_{q_j+i}-S_{q_j}|\leqslant 1-\varepsilon\}.$$

又由引理2.5.2并注意到 (2.5.9)，我们得

$$P\{\max_{1\leqslant i\leqslant\theta^k}\gamma'_{q_jk}|S_{q_j+i}-S_{q_j}|\leqslant 1-\varepsilon\} \tag{2.5.13}$$

$$\leqslant \exp\left\{-\frac{\pi^2}{8(1-\varepsilon)}\sigma^2_{q_j,(\theta^k)}r'^2_{q_jk}\right\}$$

$$\leqslant \exp\{-(1-\varepsilon)^{-1/2}(\log(q_j \vee \theta^{k+1})/\theta^{k+1}+\log\log\theta^{k+1})\}$$

$$\leqslant\{((q_j\vee\theta^{k+1})/\theta^{k+1})\log\theta^{k+1}\}^{-(1-\varepsilon)^{-1/2}}.$$

由(2.5.11),(2.5.12)和 (2.5.13), 存在常数$C(\theta)>0$和$C(\theta,\varepsilon)>0$,我们有

$$\sum_{k=1}^{\infty}P\{\min_{0\leqslant n\leqslant M_k}\max_{1\leqslant i\leqslant\theta^k}r'_{nk}|S_{n+i}-S_n|\leqslant 1-3\varepsilon\}$$

$$\leqslant 2\sum_{k=1}^{\infty}\sum_{j=0}^{m_k-1}\{((q_j\vee\theta^{k+1})/\theta^{k+1})\log\theta^{k+1}\}^{-(1-\varepsilon)^{-1/2}}$$

$$\leqslant C(\theta)\sum_{k=1}^{\infty}\left\{(\log k)k^{-(1-\varepsilon)^{-1/2}}+\sum_{j=0}^{m_k-1}(jk/\log(jk))^{-(1-\varepsilon)^{-1/2}}\right\}$$

$$\leqslant 2C(\theta)\left\{\sum_{k=1}^{\infty}((\log k)/k)^{(1-\varepsilon)^{-1/2}}\right\}^2$$

$$\leqslant C(\theta,\varepsilon).$$

由此可得

$$\varliminf_{k\to\infty}\min_{0\leqslant n\leqslant M_k}\max_{1\leqslant i\leqslant\theta^k}\gamma'_{nk}|S_{n+i}-S_n|\geqslant 1-3\varepsilon \qquad \text{a.s.}$$

由 (2.5.10) 得证 (2.5.7) 成立。

**第二步** 对任给$0<\varepsilon<1/4$,我们有

$$(2.5.14)\quad \varlimsup_{N\to\infty}\max_{0\leqslant k\leqslant a_N}\gamma_{NN}|S_{N+k}-S_N|\leqslant 1/(1-\varepsilon) \qquad \text{a.s.}$$

**证** 设$N_1=1$.对$k\geqslant 1$令$N_{k+1}=N_k+a_{N_k}$.再由引理 2.5.2, 我们得

$$(2.5.15)\ P\{\max_{0\leqslant n\leqslant a_{N_k}}\gamma_{N_kN_k}|S_{N_k+n}-S_{N_k}|\leqslant 1/(1-\varepsilon)\}$$

$$\geqslant \exp\{-(1-\varepsilon)(\log N_k/a_{N_k}+\log\log N_k)\}$$

$$\geqslant (N_{k+1}-N_k)/(N_k\log N_k),$$

由于$\max\limits_{0\leqslant n\leqslant a_{N_k}}\gamma_{N_kN_k}|S_{N_k+n}-S_{N_k}|$, $k=1,2,\cdots$是独立的且

$\sum_{k=1}^{\infty}(N_{k+1}-N_k)/(N_k\log N_k)=\infty$，按Borel-Cantelli引理就得

$$(2.5.16)\quad \varliminf_{k\to\infty}\max_{0\leqslant n\leqslant a_{N_k}}\gamma_{N_kN_k}|S_{N_k+n}-S_{N_k}|\leqslant 1/(1-\varepsilon)\quad \text{a.s.}$$

这就证明了(2.5.14)。

**第三步** 若条件(iii)被满足，那么对任给的$0<\varepsilon<1/4$，我们有

$$(2.5.17)\quad \varlimsup_{N\to\infty}\min_{0\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n+k}-S_n|\leqslant 1+\varepsilon\quad \text{a.s.}$$

**证** 容易看出由条件(iii)有$\lim\limits_{N\to\infty}N/a_N=\infty$，记$p:=p_N=[N/a_N]$.那么由引理2.5.2对充分大的$N$有

$$(2.5.18)\quad P\{\min_{0\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n+k}-S_n|\geqslant 1+\varepsilon\}$$

$$\leqslant P\{\min_{0\leqslant j\leqslant p}\max_{1\leqslant k\leqslant a_N}\gamma_{ja_N,N}|S_{ja_N+k}-S_{ja_N}|\geqslant 1+\varepsilon\}$$

$$=\prod_{j=0}^{p}\{1-P(\max_{1\leqslant k\leqslant a_N}\gamma_{ja_N,N}|S_{ja_N+k}-S_{ja_N}|<1+\varepsilon)\}$$

$$\leqslant\prod_{j=0}^{p}\left\{1-\exp\left(-\frac{1}{1+\varepsilon}\log\frac{N\log N}{a_N}\right)\right\}$$

$$\leqslant\prod_{j=0}^{p}\{1-(a_N/N)^{1/(1+\varepsilon)}/\log N\}$$

$$\leqslant\exp\left\{-\sum_{j=0}^{p}(a_N/N)^{1/(1+\varepsilon)}/\log N\right\}$$

$$\leqslant\exp\{-(N/a_N)^{\varepsilon/2}/\log N\}$$

$$\leqslant N^{-2},$$

其中最后一不等式是由条件(iii)推出的.按Borel-Cantelli引理得证(2.5.17)成立.这就完成了定理2.5.1的证明.

由定理2.5.1即可写出

**推论2.5.1**(邵启满，1990) 设$\{X_n\}$是独立随机变量序列，

$EX_n=0$.假设$\{X_n^2,n\geqslant 1\}$一致可积且

$$D_N=\sum_{i=1}^{N}EX_i^2\to\infty\quad(N\to\infty),$$

则有

$$(2.5.19)\qquad \varlimsup_{N\to\infty}\left(\frac{8\log\log D_N}{\pi^2 D_N}\right)^{1/2}\max_{1\leqslant n\leqslant N}|S_n|=1\qquad \text{a.s.}$$

**注2.5.1** 在那启满（1990）中还给出了较定理2.5.1更为一般的结论.在那里还给出了使得钟开莱重对数律成立，但重对数律不成立的例子，这就是

**例2.5.1** 设独立随机变量序列$\{X_n,n\geqslant 1\}$具有分布

$$P\{X_n=n^{1/2}\log\log n\}=P\{X_n=-n^{1/2}\log\log n\}$$
$$=(2n(\log\log n)^3)^{-1},$$

$$P\{X_n=2\}=P\{X_n=-2\}=\frac{1}{8}-\frac{1}{8\log\log n},$$

$$P\{X_n=0\}=\frac{3}{4}-\frac{1}{n(\log\log n)^3}+\frac{1}{4\log\log n}.$$

此时$EX_n=0$，$EX_n^2=1$，且$\{X_n^2,n\geqslant 1\}$一致可积，所以由推论2.5.1可知（2.5.19）成立，即

$$\varliminf_{N\to\infty}\max_{1\leqslant n\leqslant N}\left(\frac{8\log\log N}{\pi^2 N}\right)^{1/2}\left|\sum_{i=1}^{n}X_i\right|=1\qquad \text{a.s.}$$

但对任一$C>0$有

$$\sum_{n=1}^{\infty}P\{|X_n|\geqslant C(n\log\log n)^{1/2}\}=\infty,$$

所以

$$\varlimsup_{N\to\infty}\left|\sum_{i=1}^{N}X_i\right|\Big/(2N\log\log N)^{1/2}=\infty\qquad \text{a.s.}$$

这就是说，即使在重对数律不成立的情况下，钟的重对数律仍可以成立.但我们不知道是否存在独立随机变量序列，它满足重对数律而不满足钟重对数律.我们也还不知道钟重对数律成立的必

要条件是怎样的。

## § 2.6 借助强逼近对部分和的研究

在前几节中，通过直接估计部分和的概率，对部分和的增量建立了若干结论。另一方面，利用强逼近我们可容易地得到关于i.i.d.随机变量部分和的增量的性质（参见Csörgő和Révész,(1981)）。在本节中，我们首先从Sakhanenko(1984)的结果导出关于独立随机变量的部分和的强逼近定理，然后，再考察部分和的增量。

### 2.6.1 Sakhanenko定理

强逼近定理是现代概率论杰出成就之一。它是说随机变量的部分和可用一个Wiener过程逼近它。如果逼近的误差充分地小，那么许多对Wiener过程易于证明的已知的极限定理对于部分和将继续成立。关于i.i.d.随机变量，最佳可能结果是属于Komlós,Major和Tusnády（1975，1976）的。

**定理2.6.1** (Komlós, Major and Tusnády, 1975, 1976)设$\{X_n, n\geqslant 1\}$是i.i.d.随机变量序列具有$EX_1=0, EX_1^2=1, E|X_1|^p<\infty, p>2$。那么我们可在一个较大的概率空间中重新定义$\{X_n, n\geqslant 1\}$，在其上有一个Wiener过程$W$使得

$$\sum_{i=1}^{n} X_i - W(n) = o(n^{1/p}) \quad \text{a.s.} \tag{2.6.1}$$

对强逼近的一般化结论及其应用，读者可参见Csörgő-Révész（1981）。

关于独立非同分布随机变量，Sakhanenko（1984）建立了如下深刻的结果。

**定理2.6.2** (Sakhanenko, 1984)　设$\{\xi_n;\ n\geqslant 1\}$是独立随机变量序列具有$E\xi_n=0$，且对任一$j\geqslant 1$，，对某一$\lambda>0$

$$\lambda E|\xi_j|^3 e^{\lambda|\xi_j|} \leqslant E\xi_j^2. \tag{2.6.2}$$

那么可在一个其上定义有正态随机变量序列 $\{\eta_n; n\geqslant 1\}$，$\eta_n\sim N(0,\ \mathrm{Var}\xi_n)$，的较大的概率空间上重新定义$\{\xi_n; n\geqslant 1\}$，使得

$$(2.6.3)\quad E\exp\left\{C\lambda \max_{i\leqslant n}\left|\sum_{j=1}^{i}\xi_j-\sum_{j=1}^{i}\eta_j\right|\right\}\leqslant 1+\lambda\sum_{i=1}^{n}\mathrm{Var}\xi_i,$$

其中$C$是绝对常数.

定理2.6.2的证明十分繁琐，限于篇幅不在此陈述了.从定理2.6.2，我们有

**定理2.6.3** 假设（2.6.2）被满足且

$$\sum_{i=1}^{n}E\xi_i^2\leqslant B_n\to\infty\quad n\to\infty,$$

其中$\{B_n\}$是一正数序列，那么我们有

$$(2.6.4)\qquad \left|\sum_{i=1}^{n}\xi_i-\sum_{i=1}^{n}\eta_i\right|\leqslant\frac{1}{\lambda C}\log B_n\qquad \text{a.s.}$$

特别地，若 $\sum\limits_{i=1}^{n}E\xi_i^2\to\infty$，那么

$$(2.6.5)\quad \left|\sum_{i=1}^{n}\xi_i-\sum_{i=1}^{n}\eta_i\right|\leqslant\frac{1}{\lambda C}\log\left(\sum_{i=1}^{n}\mathrm{Var}\xi_i\right)\qquad \text{a.s.}$$

**证** 令$A_k=\{n: 2^k<B_n\leqslant 2^{k+1}\}$，$\mathscr{A}=\{k: A_k\neq\phi\}$.注意到$B_n\to\infty$，故$\mathscr{A}$中有无穷多个数.写$\mathscr{A}=\{k_1,k_2,\cdots\}$，$n_i=\max\{m: m\in A_{k_i}\}$.那么对每一$\varepsilon>0$

$$P\left\{\max_{i\leqslant n_k}\left|\sum_{j=1}^{i}\xi_j-\sum_{j=1}^{i}\eta_j\right|\geqslant\frac{1+\varepsilon}{\lambda C}\log B_{n_k}\right\}$$

$$\leqslant\exp\{-(1+\varepsilon)\log B_{n_k}\}$$

$$\times E\exp\left\{\lambda C\max_{i\leqslant n_k}\left|\sum_{j=1}^{i}\xi_j-\sum_{j=1}^{i}\eta_j\right|\right\}$$

$$\leqslant\exp\{-(1+\varepsilon)\log B_{n_k}\}\cdot\left(1+\lambda\sum_{i=1}^{n_k}\mathrm{Var}\xi_i\right)$$

$$\leqslant(1+\lambda B_{n_k})\exp\{-(1+\varepsilon)\log B_{n_k}\}$$
$$\leqslant(1+\lambda B_{n_k})/B_{n_k}^{1+\varepsilon}$$
$$\leqslant(1+\lambda)2^{-k\varepsilon}.$$

这样，从Borel-Cantelli引理即得

$$(2.6.6)\quad \max_{i\leqslant n_k}\left|\sum_{j=1}^{i}\xi_j-\sum_{j=1}^{i}\eta_j\right|\leqslant\frac{1+\varepsilon}{\lambda C}\log B_{n_k}\qquad \text{a.s.}$$

对任一$m\geqslant1$，我们可求得一个整数$k_i$使得$m\in A_{k_i}$，因此，当$m$充分大时

$$(2.6.7)\quad \max_{j\leqslant m}\frac{\left|\sum_{l=1}^{j}\xi_l-\sum_{l=1}^{j}\eta_l\right|}{\log B_m}\leqslant\max_{j\leqslant n_i}\frac{\left|\sum_{l=1}^{j}\xi_l-\sum_{l=1}^{j}\eta_l\right|}{\log 2^{k_i}}$$

$$\leqslant\frac{\log 2^{k_i+1}}{\log 2^{k_i}}\cdot\frac{\max_{j\leqslant n_i}\left|\sum_{l=1}^{j}\xi_l-\sum_{l=1}^{j}\eta_l\right|}{\log B_{n_i}}$$

$$\leqslant\frac{1+\varepsilon}{\lambda C}\frac{\log 2^{k_i+1}}{\log 2^{k_i}}$$

$$\leqslant\frac{1+2\varepsilon}{\lambda C}\qquad \text{a.s.}$$

由(2.6.7)及$\varepsilon$的任意性得证(2.6.4)。

**定理2.6.4** 设$\{H(n);n\geqslant1\}$是正数的不减序列且$\{X_n;n\geqslant1\}$是独立随机变量序列，$EX_n=0,EX_n^2<\infty$.假设存在$0<\alpha<\delta\leqslant1,C>0$使得

(2.6.8)对任一$\varepsilon>0$，$\sum_{n=1}^{\infty}P\{|X_n|\geqslant\varepsilon H(n)\}<\infty$，

(2.6.9)对每一$n\geqslant1$，$E|X_n|^{2+\delta}\leqslant CH(n)^{\delta-\alpha}EX_n^2$，

(2.6.10)对每一$n\geqslant1$，$n^\alpha\leqslant CH(n)$.

那么我们可在一个其上定义有正态随机变量序列$\{Y_n;n\geqslant1\}$，$Y_n\sim N(0,\ \mathrm{Var}X_nI(|X_n|\leqslant H(n)))$的较大的概率空间上重新定义$\{X_n;n\geqslant1\}$，使得

(2.6.11) $\left|\sum_{i=1}^{n}(X_i - EX_iI(|X_i| \leqslant H(i))) - \sum_{i=1}^{n} Y_i\right| = o(H(n))$ a.s.

**证** 由 (2.6.8)，存在$\varepsilon_n \to 0$满足$\varepsilon_n \log n \to \infty$使得

(2.6.12) $$\sum_{n=1}^{\infty} P\{|X_n| \geqslant \varepsilon_n H(n)\} < \infty.$$

令 $\overline{X}_n = X_nI(|X_n| \leqslant \varepsilon_n H(n)) - EX_nI(|X_n| \leqslant \varepsilon_n H(n))$，$a_n = \max_{i \leqslant n} \varepsilon_i H(i)/\log H(i)$，$\xi_n = \overline{X}_n/a_n$。首先我们来证明$\{\xi_n\}$满足(2.6.2)式，这只需证明对$0<\lambda<a/16$

(2.6.13) $$\lambda E e^{\lambda|\overline{X}_n|/a_n}|\overline{X}_n|^3/a_n \leqslant E\overline{X}_n^2.$$

注意到在 $(0,\infty)$ 上$x^\delta e^x$是增函数且$a_n \geqslant \varepsilon_n H(n)/\log H(n)$，由(2.6.9)和(2.6.10)，当$n$充分大时我们有

(2.6.14) $$\begin{aligned}\lambda E e^{\lambda|\overline{X}_n|/a_n}|\overline{X}_n|^3/a_n &\leqslant \lambda^\delta E e^{2\lambda|\overline{X}_n|/a_n}|\overline{X}_n|^{2+\delta}/a_n^\delta \\ &\leqslant \lambda^\delta E e^{2\lambda|\overline{X}_n|(\log H(n))/\varepsilon_n H(n)}|\overline{X}_n|^{2+\delta}/a_n^\delta \\ &\leqslant 9\lambda^\delta e^{4\lambda \log H(n)} E|\overline{X}_n|^{2+\delta}/a_n^\delta \\ &\leqslant 9\lambda^\delta H(n)^{4\lambda}(\log H(n))^\delta E|X_n|^{2+\delta}/(\varepsilon_n H(n))^\delta \\ &\leqslant 9C\lambda^\delta H(n)^{-a/2} EX_n^2 \\ &\leqslant EX_n^2/2.\end{aligned}$$

另一方面，又有

(2.6.15) $$\begin{aligned}E\overline{X}_n^2 &= EX_n^2 I(|X_n| \leqslant \varepsilon_n H(n)) - (EX_nI(|X_n| > \varepsilon_n H(n)))^2 \\ &\geqslant EX_n^2 - 2EX_n^2 I(|X_n| > \varepsilon_n H(n)) \\ &\geqslant EX_n^2 - 2E|X_n|^{2+\delta}/(\varepsilon_n H(n))^\delta \\ &\geqslant EX_n^2 - 2CH(n)^{\delta-a} EX_n^2/(\varepsilon_n H(n))^\delta \\ &\geqslant EX_n^2(1 - 2C/\varepsilon_n^\delta H(n)^a) \geqslant EX_n^2/2.\end{aligned}$$

综合(2.6.14)与(2.6.15)得证(2.6.13)成立。由定理2.6.3我们可在一个定义有独立正态随机变量序列$\{\eta_n; n \geqslant 1\}$，$\eta_n \sim N(0, \mathrm{Var}\xi_n)$，的较大的概率空间上重新定义$\{\xi_n; n \geqslant 1\}$使得

(2.6.16) $$\left|\sum_{i=1}^{n}\xi_i - \sum_{i=1}^{n}\eta_i\right| \leqslant \frac{16}{aC}\log\left(\sum_{i=1}^{n}\mathrm{Var}\xi_i + H(n)\right) \text{ a.s.}$$

由 (2.6.9)，我们得

$$(EX_n^2)^{\frac{2+\delta}{2}} \leqslant E|X_n|^{2+\delta} \leqslant CH(n)^{\delta-\alpha} EX_n^2.$$

即

$$EX_n^2 \leqslant C^{2/\delta} H(n)^{2-2\alpha/\delta}.$$

因此，由 (2.6.10)

$$\begin{aligned}
\sum_{i=1}^{n} \mathrm{Var}\xi_i + H(n) &\leqslant 2\sum_{i=1}^{n} EX_i^2/a_i^2 + H(n) \\
&\leqslant 2\sum_{i=1}^{n} (EX_i^2/H(i)^2)\log^2 H(i) + H(n) \\
&\leqslant 2nC^{2/\delta} H(n)^2 + H(n) \\
&\leqslant 3C^{2/\delta+1/\alpha} H(n)^{2+1/\alpha},
\end{aligned}$$

结合 (2.6.16) 推得

(2.6.17) $$\left|\sum_{i=1}^{n}\xi_i - \sum_{i=1}^{n}\eta_i\right| \leqslant \frac{16}{\alpha C}\left(2+\frac{1}{\alpha}\right)\log H(n) \quad \text{a.s.}$$

令$\overline{S}_n = \sum_{i=1}^{n}\overline{X}_i$，$T_n = \sum_{i=1}^{n}\overline{X}_i/a_i = \sum_{i=1}^{n}\xi_i$，$\overline{U}_n = \sum_{i=1}^{n}a_i\eta_i$，$V_n = \sum_{i=1}^{n}\eta_i$.

那么

$$\begin{aligned}
|\overline{S}_n - \overline{U}_n| &= \left|\sum_{i=1}^{n}a_i(T_i - T_{i-1}) - \sum_{i=1}^{n}a_i(V_i - V_{i-1})\right| \\
&= \left|a_n(T_n - V_n) - \sum_{i=1}^{n-1}(T_i - V_i)(a_{i+1} - a_i)\right| \\
&\leqslant a_n|T_n - V_n| + a_n \max_{1\leqslant i\leqslant n}|T_i - V_i| \\
&= a_n\frac{32}{\alpha C}\left(2+\frac{1}{\alpha}\right)\log H(n) \qquad \text{a.s.}
\end{aligned}$$

易见$a_n\log H(n) = o(H(n))$.故得

(2.6.18) $$|\overline{S}_n - \overline{U}_n| = o(H(n)) \qquad \text{a.s.}$$

设$\overline{Y}_n = a_n\eta_n$，那么$\{\overline{Y}_n; n\geqslant 1\}$是独立正态随机变量，$\overline{Y}_n \sim N(0, \mathrm{Var}X_nI(|X_n|\leqslant \varepsilon_n H(n)))$.为完成我们的证明，令

$$Y_n=\frac{(\mathrm{Var}X_nI(|X_n|\leqslant H(n)))^{1/2}}{(\mathrm{Var}X_nI(|X_n|\leqslant\varepsilon_nH(n)))^{1/2}}\bar{Y}_n.$$

显然，$\{Y_n;n\geqslant1\}$是独立正态随机变量，$Y_n\sim N(0,\mathrm{Var}X_nI(|X_n|\leqslant H(n)))$.那么由（2.6.12）我们有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mathrm{Var}(Y_n-\bar{Y}_n)}{H(n)^2}&=\sum_{n=1}^{\infty}\frac{1}{H(n)^2}\{(\mathrm{Var}X_nI(|X_n|\leqslant H(n)))^{1/2}\\&\quad-(\mathrm{Var}X_nI(|X_n|\leqslant\varepsilon_nH(n)))^{1/2}\}^2\\&\leqslant\sum_{n=1}^{\infty}\frac{1}{H(n)^2}|\mathrm{Var}X_nI(|X_n|\leqslant H(n))\\&\quad-\mathrm{Var}X_nI(|X_n|\leqslant\varepsilon_nH(n))|\\&\leqslant\sum_{n=1}^{\infty}\frac{1}{H(n)^2}\{EX_n^2I(\varepsilon_nH(n)<|X_n|\\&\leqslant H(n))-(EX_nI(|X_n|<H(n)))^2-(EX_nI(|X_n|\\&<\varepsilon_nH(n)))^2\}\\&\leqslant\sum_{n=1}^{\infty}\frac{3}{H(n)^2}H(n)^2P\{|X_n|\geqslant\varepsilon_nH(n)\}<\infty.\end{aligned}$$

所以由熟知的Kolmogorov强大数律，我们有

$$(2.6.19)\qquad |\sum_{i=1}^{n}(Y_i-\bar{Y}_i)|=o(H(n))\qquad \text{a.s.}$$

再利用（2.6.12），我们有

$$(2.6.20)\ |\sum_{i=1}^{n}EX_iI(|X_i|\leqslant H(i))-\sum_{i=1}^{n}EX_iI(|X_i|\leqslant\varepsilon_iH(i))|=o(H(n))\qquad \text{a.s.}$$

和

$$(2.6.21)\qquad P(X_n\neq X_nI(|X_n|\leqslant\varepsilon_nH(n))\quad \text{i.o.})=0.$$

从（2.6.18）—（2.6.21）即得待证的

$$(2.6.22)\ |\sum_{i=1}^{n}(X_i-EX_iI(|X_i|\leqslant H(i)))-\sum_{i=1}^{n}Y_i|=o(H(n))\ \text{a.s.}$$

定理2.6.4证毕.

从定理2.6.4即可看出我们还有

**定理2.6.5** 设$\{H(n);n\geq 1\}$是不减的正数序列且$\{X_n;n\geq 1\}$是独立随机变量序列,$EX_n=0$,$EX_n^2<\infty$.假设存在$0<\delta\leq 1$,$\theta\geq 0$,$C_1,C_2>0$,$\alpha>0$使(2.6.8)被满足,且满足:对每一$n\geq 1$

$$C_1 n^{\theta}\leq EX_n^2\leq (E|X_n|^{2+\delta})^{\frac{2}{2+\delta}}\leq C_2 n^{\theta}, \tag{2.6.23}$$

$$H(n)\geq C_1 n^{\theta/2+\alpha}. \tag{2.6.24}$$

那么(2.6.11)成立.

### 2.6.2 部分和的增量有多大?

在讨论应用定理2.6.4和2.6.5研究独立不同分布随机变量部分和的增量之前,我们先按如下方式重述定理1.1.4.

**定理2.6.6.**(邵启满,1989) 设$\{a_N; N\geq 1\}$,$\{b_N;N\geq 1\}$是非负整数序列,$\{Y_n;n\geq 1\}$是独立正态随机变量序列,$Y_n\sim N(0,\sigma_n^2)$.令

$$\sigma_{k,N}^2=\sum_{i=k+1}^{k+N}\sigma_i^2,$$

$$\beta_{k,n}=\{2\sigma_{k,n}^2(\log\sigma_{0,n+k}^2/\sigma_{k,n}^2+\log\log\sigma_{k,n}^2)\}^{-1/2},$$

$$\alpha_{n,N}=\left\{2\sigma_{n,a_N}^2\left(\log\frac{\sigma_{0,a_N+b_N}^2}{\sigma_{n,a_N}^2}+\log\log\sigma_{0,a_N+b_N}^2\right)\right\}^{-1/2}.$$

假设存在常数$A>0$使对每一$N\geq 2$

$$\sum_{i=b_{N-1}+1}^{b_N}\sigma_i^2\leq A\sum_{i=b_N+1}^{a_N+b_N}\sigma_i^2, \tag{2.6.25}$$

$$\sum_{i=1}^{a_N+b_N}\sigma_i^2\leq A\sum_{i=1}^{a_{N-1}+b_{N-1}}\sigma_i^2, \tag{2.6.26}$$

$$\lim_{N\to\infty}\min_{0\leq n\leq b_N}\sigma_{n,a_N}^2=\infty. \tag{2.6.27}$$

那么

$$\varlimsup_{N\to\infty}\max_{0\leq n\leq b_N}\max_{a_N\leq k<\infty}\max_{1\leq j\leq k}\beta_{n,k}\left|\sum_{i=n+1}^{n+j}Y_i\right|=1\ \text{a.s.} \tag{2.6.28}$$

(2.6.29) $$\overline{\lim_{N\to\infty}}\alpha_{b_N,N}\left|\sum_{i=b_N+1}^{b_N+a_N}Y_i\right|=1 \quad \text{a.s.}$$

邵启满（1989）证明了下述定理。

**定理2.6.7** 设 $\{a_N;N\geqslant1\}$，$\{b_N;N\geqslant1\}$是非负整数序列且$\{H(n);n\geqslant1\}$，$\{X_n;n\geqslant1\}$如定理2.6.4，满足（2.6.8）—（2.6.10）．令

$$\sigma^{*2}_{n,k}=\sum_{i=n+1}^{n+k}EX_i^2,$$

$$\beta^*_{n,k}=\left(2\sigma^{*2}_{n,k}\left(\log\frac{\sigma^{*2}_{0,n+k}}{\sigma^{*2}_{n,k}}+\log\log\sigma^{*2}_{n,k}\right)\right)^{-1/2},$$

$$\alpha^*_{n,N}=\left\{2\sigma^{*2}_{n,a_N}\left(\log\frac{\sigma^{*2}_{0,a_N+b_N}}{\sigma^{*2}_{n,a_N}}+\log\log\sigma^{*2}_{0,a_N+b_N}\right)\right\}^{-1/2}.$$

假设存在一个常数$A>0$使对每一N≥2有

(2.6.30) $$\sum_{i=b_{N-1}+1}^{b_N}EX_i^2\leqslant A\sum_{i=b_N+1}^{a_N+b_N}EX_i^2,$$

(2.6.31) $$\sum_{i=1}^{a_N+b_N}EX_i^2\leqslant A\sum_{i=1}^{a_{N-1}+b_{N-1}}EX_i^2,$$

且对$0\leqslant n\leqslant b_N$和任一$N\geqslant1$有

(2.6.32) $$H^2(n+a_N)\leqslant A\sigma^{*2}_{n,a_N}\left(\log\frac{\sigma^{*2}_{0,n+a_N}}{\sigma^{*2}_{n,a_N}}+\log\log\sigma^{*2}_{n,a_N}\right),$$

那么，我们有

(2.6.33) $$\overline{\lim_{N\to\infty}}\max_{0\leqslant n\leqslant b_N}\max_{1\leqslant j\leqslant a_N}\beta^*_{n,a_N}\left|\sum_{i=n+1}^{n+j}(X_i-EX_iI(|X_i|\leqslant H(i)))\right|=1 \quad \text{a.s.}$$

(2.6.34) $$\overline{\lim_{N\to\infty}}\alpha^*_{b_N,N}\left|\sum_{i=b_N+1}^{b_N+a_N}(X_i-EX_iI(|X_i|\leqslant H(i)))\right|=1 \quad \text{a.s.}$$

**证** 设$\sigma_i^2=\mathrm{Var}X_iI(|X_i|\leqslant H(i))$．由定理2.6.4和2.6.6及(2.6.32)，我们有

(2.6.35) $\overline{\lim\limits_{N\to\infty}}\max\limits_{0\leqslant n\leqslant b_N}\max\limits_{1\leqslant j\leqslant a_N}\beta_{n,a_N}\left|\sum\limits_{i=n+1}^{n+j}(X_i\right.$

$$\left.-EX_iI(|X_i|\leqslant H(i))\right|=1\qquad\text{a.s.}$$

(2.6.36) $\overline{\lim\limits_{N\to\infty}}\alpha_{b_N,N}\left|\sum\limits_{i=b_N+1}^{b_N+a_N}(X_i-EX_iI(|X_i|\leqslant H(i)))\right|=1\quad\text{a.s.}$

为完成定理的证明，只需验证当$N\to\infty$时

(2.6.37) $\beta_{n,a_N}/\beta^*_{n,a_N}\to1$ 关于$0\leqslant n\leqslant b_N$一致地成立，

(2.6.38) $$\alpha_{b_N,N}/\alpha^*_{b_N,N}\to1.$$

从 (2.6.9) 和 (2.6.10) 可推得

$$\begin{aligned}EX_i^2&\geqslant \mathrm{Var}X_iI(|X_i|\leqslant H(i))=\sigma_i^2\\&\geqslant EX_i^2-2EX_i^2I(|X_i|>H(i))\\&\geqslant EX_i^2-2E|X_i|^{2+\delta}/H(i)^\delta\\&\geqslant EX_i^2-2CEX_i^2H(i)^{\delta-\alpha}/H(i)^\delta\\&=EX_i^2(1-2C/H(i)^\alpha)\\&\geqslant EX_i^2(1-2C^{1+\alpha}/i^{\alpha^2}).\end{aligned}$$

从这些不等式，我们可推得(2.6.37)和(2.6.38)成立．定理 2.6.7 证毕．

作为定理2.6.7的推论，我们可写出下述结论，它的证明留给读者．

**定理2.6.8** 设$\{H(n);\ n\geqslant1\}$是不减的正数序列，$\{a_n;\ n\geqslant1\}$，$\{b_n;\ n\geqslant1\}$是非负整数序列且$\{X_n;\ n\geqslant1\}$是独立随机变量序列，$EX_n=0$，$EX_n^2<\infty$．假设存在$0<\delta<1$，$\theta\geqslant0$，$C_1,C_2>0$，$\alpha>0$使得 (2.6.8)，(2.6.23) 和下列条件被满足：

(2.6.39) $$b_N-b_{N-1}\leqslant Aa_N,$$

(2.6.40) $$b_N+a_N\leqslant A(a_{N-1}+b_{N-1}),$$

(2.6.41) $$H^2(n)/n^{\theta+\alpha}\text{是不减的，}$$

(2.6.42) $$a_N\geqslant C_1\frac{H^2(b_N+a_N)}{(b_N+a_N)^\theta\left(\log\dfrac{b_N+a_N}{a_N}+\log\log(b_N+a_N)\right)}.$$

那么(2.6.33)和 (2.6.34)成立．

# 第三章　无穷维Ornstein–Uhlenbeck过程导出的过程的轨道性质

## §3.1　引　言

一个实值、平稳Gauss过程$\{X(t), -\infty<t<\infty\}$称作参数为$\gamma$和$\lambda(\gamma, \lambda>0)$的Ornstein–Uhlenbeck (O–U)过程，如果$EX(t)=0$且

(3.1.1)　$\Gamma(s,t)=EX(t)X(s)=(\gamma/\lambda)\exp(-\lambda|t-s|)$.

设$X_i(\cdot)$是参数为$\gamma_i$和$\lambda_i$ $(i=1,2,\cdots)$的相互独立的O–U过程，记无穷维O–U过程$Y(t)=\{X_1(t), \cdots, X_i(t), \cdots\}$.近20年左右，由于它具有各种不同的应用背景而被广泛地加以研究。例如它被应用于描述遭受随机力的物理现象(Dawson, 1972)，它也出现在无穷维滤波和量子束的理论中(Miyahara, 1982)，也有人用它来建立某些生物系统的模型(Dawson, 1972; Walsh, 1981)。有关的详细介绍和讨论可参看Antoniadis和Carmona (1987)。

Dawson (1972)首先把$Y(\cdot)$作为下列无穷阶随机微分方程组列的平稳解加以研究：

(3.1.2)　$dX_i(t)=-\lambda_i X_i(t)dt+(2\gamma_i)^{1/2}dW_i(t)$, $i=1,2,\cdots$

其中$\{W_i(t), -\infty<t<\infty\}$是相互独立的Wiener过程.若设$\Gamma_0=\sum_{i=1}^{\infty}\gamma_i/\lambda_i<\infty$，那么当参数$t$被固定时，$Y(t)$几乎必然(a.s.)是一个$l^2$值的O–U过程$(E\|Y(t)\|^2_{l^2}=\Gamma_0)$.当对所有的$k$，$\gamma_k=1$且对所有充分大的$i$，存在$0<c\leqslant d$和$\delta>0$，使$ci^{1+\delta}\leqslant\lambda_i\leqslant di^{1+\delta}$，Dawson (1972)证明了$Y(\cdot)$在$l^2$中是a.s.连续的。因为"坐标"O–U过程$X_k(\cdot)$是连续的，由Hilbert空间的理论可知，为了得到$Y(\cdot)$的$l^2$连续性，只需证明实值过程$\chi^2(\cdot)=\|Y(\cdot)\|^2_{l^2}$是连续的。

Iscoe和 McDonald（1986）和 Schmuland（1988b）发展了研究

后一过程的技术，在条件 $\Gamma_0<\infty$ 上附加假设 $\Gamma_2=\sum_{i=1}^{\infty}\gamma_i^2/\lambda_i<\infty$

后，证明了$\chi^2(\cdot)$是连续的，从而也得到了$Y(\cdot)$在$l^2$中的连续性。

当$\gamma_k$相当大时这个结果并不是最好的。例如，Iscoe，Marcus，McDonald，Talagrand和Zinn（1990）证明了如下的结论：如果附加于条件$\Gamma_0<\infty$，假设对某个$r>1$，$\max\limits_{k\geq 1}\gamma_k((\log\gamma_k)\vee 0)^r/(\lambda_k\vee 1)<\infty$，那么$Y(\cdot)$是a.s. $l^2$连续的。Fernique（1989）在更为一般的场合下完全解决了这一连续性问题。他的定理的一个特殊情形如下：对每一$x\in R$，记$K(x)=\{k\in N:\ \gamma_k>\lambda_k x\}$和$\lambda(x)=\sup\{\lambda_k:\ k\in K(x)\}$，那么$Y(\cdot)\in l^2$是a.s.连续的，当且仅当$\Gamma_0<\infty$且$\int([\log(\lambda(x))]\vee 0)dx<\infty$。因此（参见Fernique（1989）推论1），为使$Y(\cdot)\in l^2$是a.s.连续的，一个充分条件是

$$\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)\ (1+((\log\lambda_k)\vee 0))<\infty.$$

另一方面，$\Gamma_2$的有限性给出了较$Y(\cdot)$在$l^2$中的连续性更多的结论。利用这一条件的变化形式，Schmuland（1988a）对$l^2$中的$Y(\cdot)$（也对$\chi^2(\cdot)$）建立了Hölder连续性的各种不同的阶。在$\Gamma_2<\infty$的条件下，Csörgő和Lin（1990b）得到了$\chi^2(\cdot)$的精确的连续模；Lin（1990b）对它建立了一个对数型的定律。

另一个与$Y(\cdot)\in l^2$紧密相关的过程$X(\cdot)$是由$Y(\cdot)$产生的无穷级数：

$$X(t)=\sum_{k=1}^{\infty}X_k(t),\quad -\infty<t<\infty, \tag{3.1.3}$$

其中$X_k(\cdot)$是$Y(\cdot)$的坐标过程。显然它是一个实值、平稳、零均值的Gauss过程。因此它能够通过发展对一般的Gauss过程的研究技术加以考察。特别地，$X(\cdot)$是a.s.连续的当且仅当它满足对平稳Gauss过程的连续性的Fernique充要条件（参见Jain和

Marcus(1978)，Ⅳ.2 节，推论 2.5)，也就是说，当且仅当我们写 $E|X(t)-X(s)|^2=g^2(|t-s|)$(其中 $g(u)$ 在$u>0$上是一个增函数) 时，$g(u)/(u(\log(1/u))^{1/2})$ 在零点是可积的。利用这一条件，我们也可以对$Y(\cdot)\in l^2$ 和 $X(\cdot)$ 这两个过程加以比较。例如，当$\gamma_k=1$ ($k=1, 2,\cdots$) 时，$\Gamma_2$的有限性归结为$\Gamma_0$的有限性。因此它对$Y(\cdot)$在$l^2$中的a.s.连续性是最低要求了。然而在这一情形，Iscoe和McDonald（1986，例3，由Dawson给出）指出，当

$$\sum_{k=1}^{\infty}\lambda_k^{-1}<\infty \text{但} \sum_{k=1}^{\infty}\lambda_k^{-1}(\log\lambda_k)^{1/2}=\infty \text{（例如取}\lambda_k=k(\log k)^{3/2}\text{）时，}$$

$Y(\cdot)\in l^2$是a.s.连续的，但是$X(\cdot)$并不满足刚才提到的Fernique条件。另一方面我们有

$$E\|Y(t)-Y(s)\|_{l^2}^2=E|X(t)-X(s)|^2.$$

因此，一般来说对于实值、平稳、零均值的Gauss过程$X(\cdot)$，用以检验其a.s.连续性的Fernique的充要条件，对于$l^2$中的平稳、零均值 Gauss过程 $Y(\cdot)$ 的a.s.连续性来说也是充分的。事实上，Csáki，Csörgő，Lin和Révész(1990)证明了：如果对某个$\delta>0$，成立着$\sum_{k=1}^{\infty}\gamma_k(\log(\lambda_k\vee e))^{1+\delta}/\lambda_k<\infty$，那么Gauss 过程$Y(\cdot)$以概率1连续；同时他们也建立了 $X(\cdot)$ 的连续模性质。后一过程的连续模性质也曾被Csörgő和 Lin（1990b）研究过。如同对 $\chi^2(\cdot)$ 的研究一样，对于$X(\cdot)$ 的研究除了因为它与 $Y(\cdot)$ 有关之外，它自身就有实用上的兴趣。例如，在建立神经感应系统的数学模型中就曾利用了过程$X(\cdot)$（Walsh， 1981）。

研究$Y(\cdot)$的O-U过程序列$\{X_i(\cdot)\}$的另一种自然的途径是考察它的部分和过程$X(\cdot,\cdot)$：

$$(3.1.4)\quad X(t,n)=\sum_{k=1}^{n}X_k(t),\quad -\infty<t<\infty,\quad n=1,2,\cdots$$

的轨道性质。这是一个定义在$R\times Z^+$($Z^+$为非负整数集）上的两参数随机过程，且对一切$t\in R$，定义$X(t,0)\equiv0$。易知 $EX(t,n)=0$，

而其协方差函数为

(3.1.5) $\Gamma(m,n,s,t):=EX(s,m)X(t,n)$

$$=\sum_{k=1}^{m\wedge n}(\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|),\quad m\wedge n=1,2,\cdots.$$

当对所有的$k$, $\gamma_k\equiv\gamma,\lambda_k\equiv\lambda$ 时，(3.1.5) 简化为

$$\Gamma(m,n,s,t)=(m\wedge n)\Gamma(s,t),$$

其中$\Gamma(s,t)$ 由(3.1.1)定义.也就是说，$X(t,n)$关于$n$可以看成一Wiener过程,关于$t$可以看成一O-U过程. 因此它是通过将 i.i.d. 一参数Wiener过程求和导出两参数Wiener 单的一个类比；也是通过将 i.i.d.Brown 桥求和导出 Kiefer 过程的一个类比（参见Csörgő和Révész (1981)，第1.11和1.15两节）. $X(t,n)$ 关于$t$的平稳性和关于$n$的Wiener过程特性启示Csörgő 和林正炎(1990a) 沿着Csörgő和 Révész (1981) 中第一章的思路去研究 $X(t,n)$ 的样本轨道性质. Csörgő 和林正炎 (1988a) 得到了它的一个重对数律，邵启满 (1990) 进一步改进了这一结果.

研究$X(\cdot,\cdot)$ 的另一个动机来源于经典的 Müntz-Szász 定理 (例如参见Rudin (1966)，第15.25节).该定理指出：函数序列

(3.1.6) $\{1,x^{\lambda_1},x^{\lambda_2},\cdots\},\ 0<\lambda_1<\lambda_2<\cdots$

的一切有限的线性组合在空间$C[0,1]$（上确界模）中稠密的充要条件是$\Sigma\lambda_n^{-1}=\infty$.令$x=e^t$,那么由Müntz-Szász定理可知：$\Sigma\lambda_n^{-1}=\infty$是函数序列$\{e^{\lambda_i t}\}$在$C(-\infty,0]$中稠密，因此也是函数序列

(3.1.7) $\{e^0,\ e^{-\lambda_1 t},\ e^{-\lambda_2 t},\ \cdots\},\ 0<\lambda_1<\lambda_2<\cdots$

在 $C[0,\infty)$ 中稠密（上确界模$\|f-g\|=\sup\limits_{t>0}|f(t)-g(t)|$)的充要条件.将(3.1.7)与两参数Gauss 过程 $X(t,n)$ 的协方差结构(3.1.5)相联系可以看出，一大类平稳 Gauss 过程的轨道性质与 $X(t,n)$ 的相应性质十分类似，只要后者的参数$\{\lambda_n\}$ 趋于无穷不太快.

在本章的最后一节中，我们将引入一个比较一般的两参数Gauss过程，它是通常的，如两参数Wiener过程，Kiefer过程和

由（3.1.4）定义的过程 $X(t,n)$ 在连续参数情形的一般化.这类过程的轨道性质将被研究于这一节中.

## § 3.2 部分和过程

首先，我们来研究由(3,1.4)定义的两参数Gauss过程$\{X(t,n),-\infty<t<\infty,\ n=1,2,\cdots\}$的轨道性质.我们将建立其连续模，并给出大增量结果和重对数律.这些都基于下面的大偏差结果.

### 3.2.1 大偏差

为得到关于$\{X(t,n)\}$的增量结果和重对数律，我们需要下列大偏差的不等式.

从现在起，不等式被认为是关于充分大的$N$成立的.记

$$\sigma_N^2(h)=2\sum_{i=1}^{N}\frac{\gamma_i}{\lambda_i}(1-e^{-\lambda_i h}),\quad \sigma_N^2=\sigma_N^2(h_N),$$

其中$\{h_N\}$是正数序列，

$$\Gamma_{0N}=\sum_{i=1}^{N}\frac{\gamma_i}{\lambda_i},\quad \Gamma_{1N}=\sum_{i=1}^{N}\gamma_i,\quad \lambda_N'=\max_{1\leqslant i\leqslant N}\lambda_i.$$

**引理3.2.1** 假设存在常数 $A>0$使得

$$(3.2.1)\qquad \sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i>1/h}}\gamma_i/\lambda_i\leqslant Ah\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i\leqslant 1/h}}\gamma_i\quad 当0\leqslant h\leqslant h_N.$$

那么对每一N在（$0,h_N$）上$\sigma_N^2(h)/h^a$ 是$h$的增函数，其中

$$a=\frac{1}{3(1+A)}.$$

**证** 设$f(h)=f_N(h)=\sigma_N^2(h)/h^a$.对$0<h<h_N$我们有

$$f'(h)=h^{-a-1}\Big(-a\sum_{i=1}^{N}\frac{\gamma_i}{\lambda_i}(1-e^{-\lambda_i h})+\sum_{i=1}^{N}\gamma_i he^{-\lambda_i h}\Big)$$

$$\geqslant h^{-a-1}\Big(-a\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i\leqslant 1/h}}\gamma_i h-a\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i>1/h}}\frac{\gamma_i}{\lambda_i}+\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i\leqslant 1/h}}\gamma_i he^{-\lambda_i h}\Big)$$

$$\geqslant h^{-a-1}\Big(-a(1+A)h\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i\approx 1/h}}\gamma_i+\frac{1}{3}h\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i\leqslant 1/h}}\gamma_i\Big)\geqslant 0,$$

即得引理的结论成立。

**引理3.2.2** 设$\{T_N\}$和$\{h_N\}$是正数序列，$h_N\leqslant T_N$。假设条件(3.2.1)被满足。那么对任一$\varepsilon>0$，存在常数$C=C(\varepsilon)>0$使得对任一$v>0$成立着不等式

$$(3.2.2)\quad P\{\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|\geqslant v\sigma_N\}$$

$$\leqslant(CT_N/h_N)\exp\Big(-\frac{v^2}{2+\varepsilon}\Big).$$

**证** 由引理3.2.1知对$0<h\leqslant h_N$函数$\sigma_N^2(h)/h^a$是增的，其中$a=\dfrac{1}{3(1+A)}$。设$r=r(\varepsilon)$是正数（后面确定）。令$r_1=h_N/2^r$，$t_r=[t/r_1]r_1$，我们有

$$(3.2.3)\quad |X(t+s,N)-X(t,N)|\leqslant|X((t+s)_r,N)-X(t_r,N)|$$

$$+\sum_{j=0}^{\infty}|X((t+s)_{r+j+1},N)-X((t+s)_{r+j},N)|$$

$$+\sum_{j=0}^{\infty}|X(t_{r+j+1},N)-X(t_{r+j},N)|.$$

选$r=r(\varepsilon)$充分大，成立着

$$(3.2.4)\quad P\{\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}|X((t+s)_r,N)-X(t_r,N)|$$

$$\geqslant v(1-\varepsilon/6)\sigma_N\}$$

$$\leqslant\frac{4T_Nh_N}{r_1^2}\exp\Big(-\frac{v^2}{2+\varepsilon}\Big)\leqslant(CT_N/h_N)\exp\Big(-\frac{v^2}{2+\varepsilon}\Big),$$

$$(3.2.5)\quad P\Big\{\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}\sum_{j=0}^{\infty}|X((t+s)_{r+j+1},N)$$

$$-X((t+s)_{r+j},N)|$$

$$\geqslant\sum_{j=0}^{\infty}\sigma_N((v^2+6j)/2^{a(r+j+1)})^{1/2}\Big\}$$

$$\leqslant\sum_{j=0}^{\infty}P\Big\{\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}|X((t+s)_{r+j+1},N)$$

$$-X((t+s)_{r+j},N)|$$

$$\geqslant\sigma_N((v^2+6j)/2^{\alpha(r+j+1)})^{1/2}\Big\}$$

$$\leqslant\sum_{j=0}^{\infty}(4T_N/h_N)2^{2(r+j+1)}\exp\left(-\frac{v^2+6j}{2+\varepsilon}\right)$$

$$\leqslant(CT_N/h_N)\exp\left(-\frac{v^2}{2+\varepsilon}\right),$$

类似地有

$$(3.2.6)\quad P\left\{\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}\sum_{j=0}^{\infty}|X(t_{r+j+1},N)-X(t_{r+j},N)|\right.$$

$$\left.\geqslant\sum_{j=0}^{\infty}((v^2+6j)/2^{\alpha(r+j+1)})^{1/2}\sigma_N\right\}$$

$$\leqslant(CT_N/h_N)\exp\left(-\frac{v^2}{2+\varepsilon}\right).$$

不失一般性，我们可假设$v\geqslant1$，假如 $r=r(\varepsilon)$ 充分大，那么

$$\sum_{j=0}^{\infty}\left(\frac{v^2+6j}{2^{\alpha(r+j+1)}}\right)^{1/2}$$

$$\leqslant\frac{v}{2^{\alpha r/2}}\sum_{j=0}^{\infty}\frac{1}{2^{\alpha(j+1)/2}}+\frac{1}{2^{\alpha r/2}}\sum_{j=0}^{\infty}\left(\frac{6j}{2^{\alpha(j+1)}}\right)^{1/2}\leqslant\frac{\varepsilon}{12}v.$$

综合这些不等式得证(3.2.2)式成立。

**注3.2.1** 若$h_N\leqslant\lambda_N'^{-1}$，那么条件（3.2.1）自动地被满足。进一步，我们有

**引理3.2.3** 对应于（3.2.2）我们有

$$(3.2.7)\quad P\{\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,n)-X(t,n)|\geqslant v\sigma_N\}$$

$$\leqslant(C'T_N/h_N)\exp\left(-\frac{v^2}{2+\varepsilon}\right)$$

对某 $C'=C'(\varepsilon)>0$成立，其中$v\geqslant(3(\log 2C))^{1/2}/\varepsilon$，而$C=C(\varepsilon)$

由（3.2.2）式确定.

**证** 定义

$$E_1=\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,1)-X(t,1)|\geqslant v\sigma_N\},$$

$$E_i=\{\max_{l<i}\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,l)-X(t,l)|<v\sigma_N$$

$$\leqslant\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,i)-X(t,i)|\},\quad i=2,\cdots,N.$$

对$0<\varepsilon<1$我们有

$$A_N:=\{\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,n)-X(t,n)|\geqslant v\sigma_N\}$$

$$=\bigcup_{n=1}^{N}E_n$$

$$\subset\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|\geqslant(1-\varepsilon)v\sigma_N\}$$

$$\cup\Big(\bigcup_{n=1}^{N-1}(E_n\cap\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|$$

$$<(1-\varepsilon)v\sigma_N\})\Big)$$

$$\subset\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|\geqslant(1-\varepsilon)v\sigma_N\}$$

$$\cup\Big(\bigcup_{n=1}^{N-1}(E_n\cap\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|(X(t+s,N)-X(t,N))$$

$$-(X(t+s,n)-X(t,n))|>\varepsilon v\sigma_N\})\Big).$$

注意到$\{(X(t+s,N)-X(t,N))-(X(t+s,n)-X(t,n)),$

$|t|\leqslant h_N,\ 0\leqslant s\leqslant h_N\}$和 $E_n$ 是独立的，利用（3.2.2），假如$v^2\geqslant 3(\log 2C)/\varepsilon^2$，我们有

$$P(A_N)\leqslant P\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|$$

$$\geqslant(1-\varepsilon)v\sigma_N\}$$

$$+\sum_{n=1}^{N-1}P(E_n)P\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)$$

$$-X(t,N)-(X(t+s,n)-X(t,n))|>\varepsilon v\sigma_N\}$$

$$\leqslant P\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|$$

$$\geqslant(1-\varepsilon)v\sigma_N\}$$

$$+C\sum_{n=1}^{N-1}P(E_n)\exp\Big\{-\varepsilon^2v^2\sigma_N^2/((2+\varepsilon)$$

$$\times\sum_{i=n+1}^{N}\frac{\gamma_i}{\lambda_i}(1-e^{-\lambda_i h_N}))\Big\}$$

$$\leqslant P\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|$$

$$\geqslant(1-\varepsilon)v\sigma_N\}+P(A_N)/2.$$

由此即得

$$P(A_N)\leqslant 2P\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|$$

$$\geqslant(1-\varepsilon)v\sigma_N\}.$$

从它即可推得所要证的不等式.

**引理3.2.4** 设 $\{T_N\}$，$\{h_N\}$，$\{h_N'\}$ 和 $\{h_N''\}$ 是正数序列，$h_N\leqslant T_N$ 且 $h_N'\leqslant h_N\leqslant h_N''$. 假设 $\{h_N\}$ 是不增的且条件 (3.2.1) 被满足. 那么对任给的 $\varepsilon>0$ 存在常数 $C=C(\varepsilon)>0$ 使对任一 $v>0$ 成立不等式

$$(3.2.8)\quad P\{\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N''}|X(t+s,n)-X(t,n)|$$

$$/(\sigma_{nN}(s)(v^2+(2+\varepsilon)\log\log(\sigma_{nN}^2(s)+\sigma_{nN}^{-2}(s)))^{1/2}$$

$$+\sigma_{nN}(s))\geqslant 1\}$$

$$\leqslant(CT_Nh_N''/h_N'^2)\exp\Big(-\frac{v^2}{2+\varepsilon}\Big),$$

其中 $\quad\sigma_{nN}(s)=\sigma_n(s)+\varepsilon\sigma_n(h_N')$.

**证** 设 $r=r(\varepsilon)$ 是正数，其值将在后面确定. 令 $r_1=h_N'/2^{2^r}$，且 $t_r=[t/r_1]r_1$. 此时也可写出不等式(3.2.3). 由引理3.2.1（由于 $\{h_N\}$ 不增，故对每一 $n\leqslant N$ 有 $h_N'\leqslant h_n$）容易看出，当 $r$ 充分大时

$$(E(X((t+s)_r,\ n)-X(t_r,n))^2)^{1/2}$$

$$\leqslant(E(X((t+s)_r,n)-X(t,n))^2)^{1/2}$$

$$+\ (E(X(t,n)-X(t_r,n))^2)^{1/2}$$

$$\leqslant \sigma_n(s)+\sigma_n(\gamma_1)\leqslant \sigma_{nN}(s).$$

固定$|t|\leqslant T_N$和$0\leqslant s\leqslant h_N''$.对任给$0<\varepsilon<1/2$，设$\theta=1+\varepsilon/16$，$A_k=\{n:\ n\leqslant N,\ \theta^k\leqslant\sigma_{nN}(s)<\theta^{k+1}\}$.令

$$u_n(s)=\sigma_{nN}(s)(v^2+(2+\varepsilon)\log\log(\sigma_{nN}^2(s)+\sigma_{nN}^{-2}(s))^{1/2}+\sigma_{nN}(s),$$

$$u(k)=\theta^k(v^2+(2+\varepsilon)\log\log\theta^{|2k|})^{1/2}.$$

那么，注意到$\{X((t+s)_r,n)-X(t_r,n),\ n\geqslant 1\}$是具有独立增量的序列，我们有

$$(3.2.9)\quad P\{\max_{1\leqslant n\leqslant N}|X((t+s)_r,n)-X(t_r,n)|/u_n(s)\geqslant 1-\varepsilon/8\}$$

$$\leqslant\sum_k P\{\max_{n\in A_k}|X((t+s)_r,n)-X(t_r,n)|\geqslant(1-\varepsilon/8)u(k)\}$$

$$\leqslant\sum_k\exp\left\{-\frac{1}{2}\theta^{-2}\left(1-\frac{\varepsilon}{8}\right)^2(v^2+(2+\varepsilon)\log\log\theta^{|2k|})\right\}$$

$$\leqslant c\exp\left(-\frac{v^2}{2+\varepsilon}\right).$$

由此即得

$$P\{\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N''}|X((t+s)_r,n)-X(t_r,n)|/u_n(s)$$

$$\geqslant 1-\varepsilon/8\}$$

$$\leqslant(cT_Nh_N''/h_N'^2)\exp\left(-\frac{v^2}{2+\varepsilon}\right).$$

现在我们来处理（3.2.3)右边的第一个级数，再次应用引理3.2.1我们有

$$\sigma_{nj}^2:=E(X((t+s)_{r+j+1},n)-X((t+s)_{r+j},n))^2$$

$$\leqslant\sigma_n^2(h_N'/2^{2^{r+j}})\leqslant\delta(\varepsilon)2^{-\alpha 2^j}\sigma_n^2(h_N')$$

$$\leqslant\delta(\varepsilon)2^{-\alpha 2^j}\varepsilon^{-2}\theta^{2k+2},$$

其中$\delta(\varepsilon)\leqslant 2^{-\alpha(2^r-1)}$. 取$\delta(\varepsilon)$ 充分小使得

$$\sum_{j=0}^{\infty}(\delta(\varepsilon)2^{-\alpha 2^j})^{1/2}\varepsilon^{-1}\theta^{k+1}(v^2+(2+\varepsilon)\log\log(\sigma_n^2(s)$$

$$+\sigma_n^{-2}(s))+2^{r+j+2})^{1/2}$$

$$\leqslant \frac{\varepsilon}{16} u_n(s),$$

只要$r$充分大这是可能的.那么类似于（3.2.9）得到

$$P\Big\{\max_{1\leqslant n\leqslant N}\sum_{j=0}^{\infty}|X((t+s)_{r+j+1},n)-X((t+s)_{r+j},n)|/u_n(s)$$
$$\geqslant \varepsilon/16\Big\}$$
$$\leqslant \sum_{k}\sum_{j=0}^{\infty}P\{\max_{n\in A_k}|X((t+s)_{r+j+1},n)-X((t+s)_{r+j},n)|$$
$$\geqslant (\delta(\varepsilon)2^{-\alpha 2^j})^{1/2}\varepsilon^{-1}\theta^{k+1}(v^2+(2+\varepsilon)\log\log\theta^{|2k|}+2^{r+j+2})^{1/2}\}$$
$$\leqslant \sum_{j=0}^{\infty}\sum_{k}\exp\left\{-\frac{1}{2}(v^2+(2+\varepsilon)\log\log\theta^{|2k|}+2^{r+j+2})\right\}$$
$$\leqslant c\sum_{j=0}^{\infty}\exp\left\{-\left(\frac{1}{2}v^2+2^{r+j+1}\right)\right\}.$$

由此推得

$$P\left\{\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N''}\sum_{j=0}^{\infty}|X((t+s)_{r+j+1},n)\right.$$
$$\left.-X((t+s)_{r+j},n)|/u_n(s)\geqslant\varepsilon/16\right\}$$
$$\leqslant \sum_{j=0}^{\infty}(c2^{2^{r+j+1}}T_N h_N''/h_N^{(2)})\exp\left\{-\left(\frac{1}{2}v^2+2^{r+j+1}\right)\right\}$$
$$\leqslant (cT_N h_N''/h_N^{(2)})\exp(-v^2/2).$$

对于（3.2.3)式右边第二个级数我们有类似的不等式．综合这些不等式即得所要证的（3.2.8)．

**引理3.2.5** 对任给的$\varepsilon>0$,存在常数$C=C(\varepsilon)>0$和$v(\varepsilon)>0$使对$v\geqslant v(\varepsilon)$成立着不等式

$$P\{\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}|X(t,n)|\geqslant v\Gamma_{0N}^{1/2}\} \tag{3.2.10}$$
$$\leqslant C(1+T_N\Gamma_{1N}/\Gamma_{0N})\exp\left(-\frac{v^2}{2+\varepsilon}\right).$$

**证** 我们来证对任一$v>0$有

$$(3.2.11)\qquad P\{\sup_{|t|\leqslant T_N}|X(t,N)|\geqslant v\Gamma_{1N}^{1/2}\}$$

$$\leqslant C(1+T_N\Gamma_{1N}/\Gamma_{0N})\exp\left(-\frac{v^2}{2+\varepsilon}\right).$$

由这一不等式，用类似于引理3.2.3的证明可得（3.2.10）式。

令 $d=\delta^2\Gamma_{0N}/\Gamma_{1N}$，其中$\delta=\varepsilon/64$。那么

$$(3.2.12)\ P\{\sup_{|t|\leqslant T_N}|X(t,\ N)|\geqslant v\Gamma_{0N}^{1/2}\}$$

$$\leqslant 2(1+T_N/d)P\{\sup_{0\leqslant t\leqslant d}|X(t,N)|\geqslant v\Gamma_{0N}^{1/2}\}$$

$$\leqslant 2(1+T_N/d)P\{\sup_{0\leqslant t\leqslant 1}|X(dt,N)|\geqslant v\Gamma_{0N}^{1/2}\}.$$

注意到

$$E(X(dt,N)-X(ds,N))^2=2\sum_{i=1}^{N}\frac{\gamma_i}{\lambda_i}(1-e^{-\lambda_i d|t-s|})$$

$$\leqslant 2\Gamma_{1N}d|t-s|=:\Lambda^2(|t-s|).$$

我们有

$$\int_1^{\infty}\Lambda(e^{-y^2})dy=\int_1^{\infty}(2\Gamma_{1N}de^{-y^2})^{1/2}dy\leqslant 2\delta\Gamma_{0N}^{1/2}.$$

利用Fernique引理（引理 1.5.1）我们得

$$P\{\sup_{0\leqslant t\leqslant 1}|X(dt,N)|\geqslant v\Gamma_{0N}^{1/2}\}$$

$$\leqslant P\left\{\sup_{0\leqslant t\leqslant 1}|X(dt,N)|\geqslant\frac{v}{1+8\delta}\left(\Gamma_{0N}^{1/2}+4\int_1^{\infty}\Lambda(e^{-y^2})dy\right)\right\}$$

$$\leqslant c\int_{v/(1+8\delta)}^{\infty}e^{-y^2/2}dy\leqslant c\exp\left(-\frac{v^2}{2+\varepsilon}\right).$$

代入（3.2.12）得（3.2.11）。引理证毕。

定义$\theta_N(\varepsilon)$是方程

$$(3.2.13)\qquad \sum_{i=1}^{N}(\gamma_i/\lambda_i)e^{-2\lambda_i\theta_N(\varepsilon)}=\varepsilon\Gamma_{0N}$$

的解。

**引理3.2.6** 对任给的 $0<\varepsilon<1/2$，存在常数 $C=C(\varepsilon)>0$，使当 $v\geqslant 2(\log(C(\varepsilon)(1+T_N/\theta_N(\varepsilon))))^{1/2}$ 时成立着

$$(3.2.14)\qquad P\{\sup_{0\leqslant t\leqslant T_N}|X(t,N)|\geqslant v\Gamma_{0N}^{1/2}\}$$

$$\geqslant C(1+T_N/\theta_N(\varepsilon))\exp\left(-\frac{v^2}{2(1-2\varepsilon)}\right).$$

**证** 设 $\{W_i(t),t\geqslant 0\}_{i=1}^{\infty}$ 是独立的标准 Wiener 过程序列。注意到 $\{X(t,N),\ 0\leqslant t\leqslant T_N\}$ 和 $\left\{\sum_{i=1}^{N}(\gamma_i/\lambda_i)^{1/2}W_i(e^{2\lambda_i t})/e^{\lambda_i t},\right.$ $\left.0\leqslant t\leqslant T_N\right\}$ 具有相同的分布，我们有

$$(3.2.15)\ P\{\sup_{0\leqslant t\leqslant T_N}|X(t,N)|\geqslant v\Gamma_{0N}^{1/2}\}$$

$$\geqslant P\{\max_{0\leqslant j\leqslant T_N/\theta_N}|X(j\theta_N,N)|\geqslant v\Gamma_{0N}^{1/2}\}$$

$$=P\left\{\max_{0\leqslant j\leqslant T_N/\theta_N}\left|\sum_{i=1}^{N}\left(\frac{\gamma_i}{\lambda_i}\right)^{1/2}W_i(e^{2\lambda_i j\theta_N})/e^{\lambda_i j\theta_N}\right|\right.$$

$$\left.\geqslant v\Gamma_{0N}^{1/2}\right\}.$$

令

$$U_j=\sum_{i=1}^{N}\left(\frac{\gamma_i}{\lambda_i}\right)^{1/2}W_i(e^{2\lambda_i j\theta_N})/e^{\lambda_i j\theta_N},$$

$$V_j=\sum_{i=1}^{N}\left(\frac{\gamma_i}{\lambda_i}\right)^{1/2}W_i(e^{2\lambda_i(j-1)\theta_N})/e^{\lambda_i j\theta_N}.$$

从 $\theta_N$ 的定义容易看出

$$U_j-V_j\sim N\left(0,\sum_{i=1}^{N}\frac{\gamma_i}{\lambda_i}(1-e^{-2\lambda_i\theta_N})\right)$$

$$=N(0,\ (1-\varepsilon)\Gamma_{0N}).$$

因此我们有

$$P\{\max_{0\leqslant j\leqslant T_N/\theta_N}|U_j|<v\Gamma_{0N}^{1/2}\}$$

$$=P\{\max_{0\leq j<[T_N/\theta_N]}|U_j|$$

$$<v\Gamma_{0N}^{1/2}, |U_{[T_N/\theta_N]}-V_{[T_N/\theta_N]}+V[T_N/\theta_N]|<v\Gamma_{0N}^{1/2}\}$$

$$=\int_{-\infty}^{\infty}P\{|U_{[T_N/\theta_N]}-V_{[T_N/\theta_N]}+y|<v\Gamma_{0N}^{1/2}\}dP\{V_{[T_N/\theta_N]}$$

$$<y, \max_{0\leq j<[T_N/\theta_N]}|U_j|<v\Gamma_{0N}^{1/2}\}$$

$$=\int_{-\infty}^{\infty}\left\{\Phi\left(\frac{v\Gamma_{0N}^{1/2}-y}{(1-\varepsilon)^{1/2}\Gamma_{0N}^{1/2}}\right)-\Phi\left(\frac{-v\Gamma_{0N}^{1/2}-y}{(1-\varepsilon)^{1/2}\Gamma_{0N}^{1/2}}\right)\right\}dP$$

$$\times\{V_{[T_N/\theta_N]}<y, \max_{0\leq j<[T_N/\theta_N]}|U_j|<v\Gamma_{0N}^{1/2}\}$$

$$\leq\int_{-\infty}^{\infty}\left\{\Phi\left(\frac{v}{(1-\varepsilon)^{1/2}}\right)-\Phi\left(\frac{-v}{(1-\varepsilon)^{1/2}}\right)\right\}dP$$

$$\times\{V_{[T_N/\theta_N]}<y, \max_{0\leq j<[T_N/\theta_N]}|U_j|<v\Gamma_{0N}^{1/2}\}$$

$$=\left\{1-\frac{2}{\sqrt{2\pi}}\int_{v/(1-\varepsilon)^{1/2}}^{\infty}e^{-t^2/2}dt\right\}P\{\max_{0\leq j<[T_N/\theta_N]}|U_j|<v\Gamma_{0N}^{1/2}\}$$

$$\leq(1-2Ce^{-v^2/2(1-2\varepsilon)})P\{\max_{0\leq j<[T_N/\theta_N]}|U_j|<v\Gamma_{0N}^{1/2}\},$$

这里我们利用了下列事实：

(a) $U_{[T_N/\theta_N]}-V_{[T_N/\theta_N]}$ 和 $\{V_{[T_N/\theta_N]}, U_j, 0\leq j<[T_N/\theta_N]\}$ 相互独立．

(b) 对任给 $y$ 和 $x\geq0$，$\Phi(x-y)-\Phi(-x-y)\leq\Phi(x)-\Phi(-x)$．

(c) 对任给 $\delta>0$，存在 $C(\delta)>0$ 使对任一 $x\geq0$

$$\int_x^{\infty}e^{-t^2/2}dt\geq2C(\delta)e^{-(1+\delta)x^2/2}.$$

由递推可得当 $v\geq2(\log(C(\varepsilon)(1+T_N/\theta_N))^{1/2}$ 时

(3.2.16) $P\{\max_{0\leq j\leq T_N/\theta_N}|U_j|<v\Gamma_{0N}^{1/2}\}$

$$\leqslant\{1-2C(\varepsilon)e^{-a^2/2(1-2\varepsilon)}\}^{[T_N/\theta_N]+1}$$

$$\leqslant 1-C(\varepsilon)\left(1+\frac{T_N}{\theta_N}\right)e^{-a^2/2(1-2\varepsilon)}.$$

从 (3.2.15) 和 (3.2.16) 即得 (3.2.14).

### 3.2.2 增量结果

现在我们利用上面给出的大偏差不等式来建立若干增量结果.这里将同时对充分小的 $h_N$ 和充分大的 $h_N$ 给出证明.

**定理3.2.1**(Csörgő and Lin, 1990a).设$\{T_N\}$和$\{h_N\}$是正数序列.假设$\{T_N\}$是不减的且$\{h_N\}$是单调的,条件(3.2.1)被满足,又设

(3.2.17) $\quad \lim\limits_{N\to\infty}(\log(T_N/h_N))/\log\log N=\infty.$

那么我们有

$$\lim_{N\to\infty}\sup_{|t|\leqslant T_N}a_N|X(t+h_N,N)-X(t,N)|=1 \quad \text{a.s.}$$

$$\lim_{N\to\infty}\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}a_N|X(t+s,N)-X(t,N)|=1 \quad \text{a.s.}$$

其中

$$a_N=\{2\sigma_N^2[\log T_N/h_N+\log\log(\sigma_N^2+\sigma_N^{-2})]\}^{-1/2}.$$

**注3.2.2** 若当 N→∞ 时, $h_N\to 0$, 这一定理可视作 Wiener 过程的Lévy连续模定理的一个类比(参见Csörgő和Révész(1981)的定理1.1.1和1.14.2),若当 N→∞时,$h_N\to\infty$, 这一定理可视作 Wiener 过程的大增量结果的一个类比 (参见 Csöreő和Révész (1981)定理1.2.1).

定理3.2.1的证明.

首先我们来证对$0<\varepsilon<1/2$有

(3.2.18) $$\overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}a_N|X(t+s,n)-X(t,n)|\leqslant 1+\varepsilon \quad \text{a.s.}$$

我们首先假设 $\{h_N\}$是不减的. 显然此时我们可用$a_N=\{2\sigma_N^2[\log\times T_N/h_N+\log\log\sigma_N^2]\}^{-1/2}$代替

$$\{2\sigma_N^2[\log T_N/h_N+\log\log(\sigma_N^2+\sigma_N^{-2})]\}^{-1/2}.$$

令$\theta=1+\varepsilon/2$.定义

$$H_{kj}=\{N:\ \theta^k<T_N/h_N\leqslant\theta^{k+1},\ \theta^j<\sigma_N^2\leqslant\theta^{j+1}\},$$

$$M_{kj}=\max\{N:\ N\in H_{kj}\},$$

$$\mathscr{A}=\{(k,\ j):\ H_{kj}\neq\varnothing\},\quad \sigma_{kj}'^2=\min\{\sigma_N^2:\ N\in H_{kj}\}.$$

由定义知

$$1\leqslant\sigma_{M_{kj}}^2/\sigma_{kj}'^2\leqslant\theta.$$

因此我们有

$$(3.2.19)\quad \overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}a_N|X(t+s,n)-X(t,n)|$$

$$\leqslant\overline{\lim_{\substack{k\to\infty\\ j\to\infty}}}_{(k,j)\in\mathscr{A}}\max_{N\in H_{kj}}\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}a_N|X(t+s,n)-X(t,n)|$$

$$\leqslant\overline{\lim_{\substack{k\to\infty\\ j\to\infty}}}_{(k,j)\in\mathscr{A}}\max_{1\leqslant n\leqslant M_{kj}}\sup_{|t|\leqslant T_{M_{kj}}}\sup_{0\leqslant s\leqslant h_{M_{kj}}}|X(t+s,n)-X(t,n)|/\{2\sigma_{kj}'^2\log(\theta^k\log\theta^j)\}^{1/2}$$

$$\leqslant\overline{\lim_{\substack{k\to\infty\\ j\to\infty}}}_{(k,j)\in\mathscr{A}}\max_{1\leqslant n\leqslant M_{kj}}\sup_{|t|\leqslant T_{M_{kj}}}\sup_{0\leqslant s\leqslant h_{M_{kj}}}|X(t+s,n)-X(t,n)|$$

$$/\{2\theta^{-1}\sigma_{M_{kj}}^2\log[(\theta^{-1}T_{M_{kj}}/h_{M_{kj}})\log\theta^j]\}^{1/2}.$$

利用引理3.2.3我们得

$$P\{\max_{1\leqslant n\leqslant M_{kj}}\sup_{|t|\leqslant T_{M_{kj}}}\sup_{0\leqslant s\leqslant h_{M_{kj}}}|X(t+s,n)-X(t,n)|$$

$$/\{2\theta^{-1}\sigma_{M_{kj}}^2\log[(\theta^{-1}T_{M_{kj}}/h_{M_{kj}})\log\theta^j]\}^{1/2}\geqslant1+\varepsilon\}$$

$$\leqslant(CT_{M_{kj}}/h_{M_{kj}})\exp\left\{-\frac{2\theta^{-1}(1+\varepsilon)^2}{2+\varepsilon}\log[(\theta^{-1}T_{M_{kj}}/h_{M_{kj}})\log\theta^j]\right\}$$

$$\leqslant C(T_{M_{kj}}/h_{M_{kj}})^{-\varepsilon/2}(j\log\theta)^{-1-\varepsilon/2}$$

$$\leqslant C\theta^{-k\varepsilon/2}(j\log\theta)^{-1-\varepsilon/2}.$$

这样应用 Borel-Cantelli 引理（它在两参数情形的推广是平凡

的)，从 (3.2.19) 即得 (3.2.18).

其次，我们假设$\{h_N\}$是不增的.定义

$H_k=\{N:\ \theta^k<T_N/h_N\leqslant\theta^{k+1}\}$,　　$M_k=\max\{N:\ N\in H_k\}$,

$m_k=\min\{N:\ N\in H_k\}$,　　$\mathscr{A}=\{k:\ H_k\neq\varnothing\}$.

显然有

$$1\leqslant h_{m_k}/h_{M_k}\leqslant(h_{m_k}/T_{m_k})/(h_{M_k}/T_{M_k})\leqslant\theta.$$

那么

$$(3.2.20)\quad \varlimsup_{N\to\infty}\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}\alpha_N|X(t+s,n)-X(t,n)|$$

$$\leqslant\varlimsup_{T\to\infty}\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}(1+3\varepsilon)|X(t+s,n)$$
$$-X(t,n)|/\{[(\sigma_n(s)+\varepsilon\sigma_n(h_N))(2\log T_N/h_N$$
$$+(2+\varepsilon)\log\log((\sigma_n(s)+\varepsilon\sigma_n(h_N))^2$$
$$+(\sigma_n(s)+\varepsilon\sigma_n(h_N))^{-2}))]^{1/2}+\sigma_n(s)+\varepsilon\sigma_n(h_N)\}$$

$$\leqslant\varlimsup_{k\to\infty,k\in\mathscr{A}}\max_{1\leqslant n\leqslant M_k}\sup_{|t|\leqslant T_{M_k}}\sup_{0\leqslant s\leqslant h_{m_k}}(1+3\varepsilon)|X(t+s,n)$$
$$-X(t,n)|$$
$$/\{[(\sigma_n(s)+\varepsilon\sigma_n(h_{M_k}))(2\log\theta^k+(2+\varepsilon)\log\log((\sigma_n(s)$$
$$+\varepsilon\sigma_n(h_{M_k}))^2+(\sigma_n(s)+\varepsilon\sigma_n(h_{M_k}))^{-2}))]^{1/2}+\sigma_n(s)$$
$$+\varepsilon\sigma_n(h_{M_k})\}.$$

利用引理3.2.4我们得到

$$P\{\max_{1\leqslant n\leqslant M_k}\sup_{|t|\leqslant T_{M_k}}\sup_{0\leqslant s\leqslant h_{m_k}}|X(t+s,n)-X(t,n)|$$
$$/\{[(\sigma_n(s)+\varepsilon\sigma_n(h_{M_k}))(2\log\theta^k+(2+\varepsilon)\log\log((\sigma_n(s)$$
$$+\varepsilon\sigma_n(h_{M_k}))^2$$
$$+(\sigma_n(s)+\varepsilon\sigma_n(h_{M_k}))^{-2}))]^{1/2}+\sigma_n(s)+\varepsilon\sigma_n(h_{M_k})\}\geqslant1+\varepsilon\}$$

$$\leqslant(CT_{M_k}h_{M_k}/h_{M_k}^2)\exp\left\{-\frac{2(1+\varepsilon)^2}{2+\varepsilon}\log\theta^k\right\}\leqslant C\theta^{-\varepsilon k}.$$

由此，从 (3.2.20) 即得 (3.2.18).

最后，我们来证

$$(3.2.21)\ \varliminf_{N\to\infty}\ \sup_{|t|\leqslant T_N} a_N|X(t+h_N,\ N)-X(t,\ N)|\geqslant 1-\varepsilon\quad \text{a.s.}$$

对整数$i$和$j$，$i<j$，

$$\begin{aligned}&E(X((i+1)h_N,N)-X(ih_N,N))(X((j+1)h_N,N)\\&\quad -X(jh_N,N))\\&=\sum_{l=1}^{N} E(X_l((i+1)h_N)-X_l(ih_N))(X_l((j+1)h_N)-X_l(jh_N))\\&=\sum_{l=1}^{N}\frac{\gamma_l}{\lambda_l}e^{-\lambda_l ih_N}(2e^{\lambda_l ih_N}-e^{\lambda_l(i-1)h_N}-e^{\lambda_l(i+1)h_N})<0.\end{aligned}$$

这样，利用Slepian引理（引理1.1.1）和过程的平稳性，我们得

$$\begin{aligned}(3.2.22)\ &P\{\max_{|k|\leqslant T_N/h_N} a_N|X((k+1)h_N,N)-X(kh_N,N)|\leqslant 1-\varepsilon\}\\&\leqslant(P\{a_N|X(h_N,N)|\leqslant 1-\varepsilon\})^{2[T_N/h_N]}\\&\leqslant\{1-(\exp(-(1-\varepsilon)\log T_N/h_N))/\\&\quad 6(\log T_N/h_N)^{1/2}\}^{2[T_N/h_N]}\\&\leqslant\exp\{-2(T_N/h_N)^{\varepsilon/2}\}\leqslant N^{-2},\end{aligned}$$

其中后一不等式是由于条件(3.2.17)．因此(3.2.21)被证明了．现在从(3.2.18)和(3.2.21)即得定理3.2.1的两个结论．

### 3.2.3 重对数律

**定理3.2.3** (Shao, 1990) 设$\{T_N\}$是不减的正数序列．假设

(i) 当$N\to\infty$时，$\Gamma_{0N}\to\infty$；

(ii) 存在常数$d\geqslant 1$使对每一$N\geqslant 1$有$\Gamma_{0,N+1}\leqslant d\Gamma_{0N}$；

(iii) 当 $N\to\infty$时$\log(T_N\Gamma_{1N}/\Gamma_{0N})=o(\log\log\Gamma_{0N})$．

那么我们有

$$\varlimsup_{N\to\infty}\ \sup_{|t|\leqslant T_N}|X(t,\ N)|/(2\Gamma_{0N}\log\log\Gamma_{0N})^{1/2}=1\qquad \text{a.s.}$$

$$\varlimsup_{N\to\infty}\ \max_{1\leqslant n\leqslant N}\ \sup_{|t|\leqslant T_N}|X(t,\ n)|/(2\Gamma_{0N}\log\log\Gamma_{0N})^{1/2}=1\qquad \text{a.s.}$$

**证** 仿照（3.2.18）的证明，应用引理3.2.5我们也可证明

$$(3.2.23)\ \overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}|X(t,\ n)|/(2\Gamma_{0N}\log\log\Gamma_{0N})^{1/2}$$

$$\leqslant 1 \qquad \text{a.s.}$$

细节从略。

我们来给出不等式

$$(3.2.24)\ \overline{\lim_{N\to\infty}}\sup_{|t|\leqslant T_N}|X(t,\ N)|/(2\Gamma_{0N}\log\log\Gamma_{0N})^{1/2}\geqslant 1-\varepsilon \qquad \text{a.s.}$$

$(0<\varepsilon<1/2)$ 的证明。令$N_1=1$，定义$N_{k+1}=\min\{n:\ \Gamma_{0n}\geqslant(10d/\varepsilon^2)^k\}$。由条件（ii），我们有

$$(3.2.25)\qquad (10d/\varepsilon^2)^k\leqslant\Gamma_{0N_{k+1}}<d(10d/\varepsilon^2)^k.$$

此外

$$(3.2.26)\quad \overline{\lim_{N\to\infty}}\sup_{|t|\leqslant T_N}|X(t,N)|/(2\Gamma_{0N}\log\log\Gamma_{0N})^{1/2}$$

$$\geqslant\overline{\lim_{k\to\infty}}\sup_{|t|\leqslant T_{N_k}}|X(t,N_k)|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2}$$

$$\geqslant\overline{\lim_{k\to\infty}}\sup_{|t|\leqslant T_{N_k}}|X(t,\ N_k)-X(t,N_{k-1})|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2}$$

$$-\overline{\lim_{k\to\infty}}\sup_{|t|\leqslant T_{N_k}}|X(t,N_{k-1})|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2}.$$

利用引理3.2.5和条件（iii），有

$$(3.2.27)\ P\{\sup_{|t|\leqslant T_{N_k}}|X(t,\ N_{k-1})|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2}\geqslant\varepsilon/2\}$$

$$\leqslant C(1+T_{N_k}\Gamma_{1N_{k-1}}/\Gamma_{0N_{k-1}})$$

$$\times\exp\{-(\varepsilon^2\Gamma_{0N_k}\log\log\Gamma_{0N_k})/5\Gamma_{0N_{k-1}}\}$$

$$\leqslant C(1+10T_{N_k}\Gamma_{1N_k}/\varepsilon^2\Gamma_{0N_k})(\log\Gamma_{0N_k})^{-2}$$

$$\leqslant ck^{-3/2}.$$

由此即得

$$(3.2.28)\ \overline{\lim_{k\to\infty}} \sup_{|t|\leqslant T_{N_k}} |X(t,\ N_{k-1})|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2}\leqslant\varepsilon/2 \quad \text{a.s.}$$

现在我们来证

$$(3.2.29)\qquad \overline{\lim_{k\to\infty}} \sup_{|t|\leqslant T_{N_k}} |X(t,N_k)-X(t,N_{k-1})|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2}\geqslant 1-\varepsilon/2 \quad \text{a.s.}$$

注意到$\Gamma_{0N_k}-\Gamma_{0N_{k-1}}\geqslant\Gamma_{0N_k}(1-\varepsilon^2/10)$．修改$\theta_N$的定义如下：定义$\theta_{N_k}(\varepsilon)$ 是方程

$$\sum_{i=N_{k-1}+1}^{N_k}(\gamma_i/\lambda_i)e^{-2\lambda_i\theta_{N_k}(\varepsilon)}=\varepsilon(\Gamma_{0N_k}-\Gamma_{0N_{k-1}})$$

的解，利用引理3.2.6我们有

$$P\{\sup_{|t|\leqslant T_{N_k}} |X(t,N_k)-X(t,\ N_{k-1})|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2}\geqslant 1-\varepsilon/2\}$$

$$\geqslant C\left(1+T_{N_k}/\theta_{N_k}\left(\frac{\varepsilon}{4}\right)\right)\exp\left\{-\left(1-\frac{\varepsilon}{2}\right)\times(\Gamma_{0N_k}\log\log\Gamma_{0N_k})/(\Gamma_{0N_k}-\Gamma_{0N_{k-1}})\right\}$$

$$\geqslant C(\log\Gamma_{0N_k})^{-(1-\varepsilon/4)}\geqslant Ck^{-(1-\varepsilon/4)}.$$

因此，从 $\sup_{|t|\leqslant T_{N_k}} |X(t,\ N_k)-X(t,\ N_{k-1})|$，$k\geqslant1$，的独立性，我们得 (3.2.29)．把 (3.2.28) 和 (3.2.29) 代入 (3.2.26) 式得(3.2.24)．结合 (3.2.23) 和 (3.2.24)即有定理3.2.3的结论．

## § 3.3 无穷级数

这一节的主要目的在于研究由 (3.1.3) 式定义的过程 $X(\cdot)$ 的样本轨道的性质．显然，当

$$\Gamma_0=\sum_{i=1}^{\infty}\gamma_i/\lambda_i<\infty \tag{3.3.1}$$

时，对每一固定的$t$，$X(t)$是一个均值为0、方差为$\Gamma_0$的a.s.有限的Gauss随机变量．然而，仅仅在$\Gamma_0<\infty$的条件下，$X(\cdot)$未必能作为$t\in R$上的a.s.连续的Gauss过程而存在．为此，我们首先来考察$X(\cdot)$的存在性和连续性．然后再来建立它的连续模结果．最后，我们将给出一个对数型的定律．

### 3.3.1 存在性和连续性

**定理3.3.1**(Csóki, Csörgő, Lin and Révész, 1990) 假设对某个$\delta>0$，

$$\sum_{k=1}^{\infty}\gamma_k(\log(\lambda_k\vee e))^{1+\delta}/\lambda_k<\infty. \tag{3.3.2}$$

那么，在任何有限区间上关于$t$一致地成立

$$X(t,n)\to X(t)\quad \text{a.s.}\quad n\to\infty.$$

定理的结论等价于：对任意的$\varepsilon>0$，$T>0$和几乎所有的$\omega\in\Omega$，存在一个正整数$n_0=n(\varepsilon,T,\omega)$，使得当$n\geqslant n_0$时

$$\sup_{|t|\leqslant T}|X(t,n,\omega)-X(t,\omega)|\leqslant\varepsilon. \tag{3.3.3}$$

由此立刻可以得到下列结论：

**推论3.3.1** 在条件(3.3.2)下，$\{X(t),\ -\infty<t<\infty\}$以概率1连续．

**定理3.3.1的证明**

我们证明(3.3.3)．为此，根据Itô-Nisio定理(1968)，只须证明：$\sup\limits_{|t|\leqslant T}|X(t,n)-X(t)|=\sup\limits_{|t|\leqslant T}\left|\sum\limits_{k=n+1}^{\infty}X_k(t)\right|$依概率收敛于0，也就是说，对任意$\varepsilon>0$，

$$\lim_{n\to\infty}P\left\{\sup_{|t|\leqslant T}\left|\sum_{k=n+1}^{\infty}X_k(t)\right|>\varepsilon\right\}=0.$$

后者又等价于：对任意的$\varepsilon>0$和$0<\eta<1$，存在$n_0=n_0(\varepsilon,\eta)$，使

得当$m>n\geqslant n_0$时，

(3.3.4) $$P\{\sup_{|t|\leqslant T}|X_{mn}(t)|>\varepsilon\}\leqslant\eta,$$

其中 $X_{mn}(t)=\sum_{k=n+1}^{m}X_k(t)$，它是一个平稳的、均值为 0 的 Gauss 过程，

$$EX_{mn}^2(t)=\sum_{k=n+1}^{m}\gamma_k/\lambda_k.$$

$$EX_{mn}(t)X_{mn}(s)=\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|),$$

$$E(X_{mn}(t)-X_{mn}(s))^2$$

$$=2\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)(1-\exp(-\lambda_k|t-s|)).$$

现在我们将应用Fernique引理(引理1.5.1)于过程$X_{mn}(t)$，$t\in[-T,T]$，其中$n<m$和$T>0$是固定的。为了能应用这个引理，我们首先证明：在条件 (3.3.2) 下，成立着

(3.3.5) $$\int_0^{1/e}\frac{\left(\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)(1-\exp(-\lambda_k u))\right)^{1/2}}{u(\log(1/u))^{1/2}}du<\infty.$$

此时这一积分的有限性等价于对于

$$A(u)=\left(2\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)(1-\exp(-\lambda_k u))\right)^{1/2},$$

引理1.5.1中的积分

$$\int_1^{\infty}A(e^{-y^2})dy<\infty.$$

记$K_1=\{k:\lambda_k<u^{-1/2}\}\cap[n+1,m]$，$K_2=\{k:\lambda_k\geqslant u^{-1/2}\}\cap[n+1,m]$。那么有

当$k\in K_1$时，$(\gamma_k/\lambda_k)(1-\exp(-\lambda_k u))\leqslant(\gamma_k/\lambda_k)u^{1/2}$，

当$k\in K_2$时，$\frac{1}{2}\log\frac{1}{u}\leqslant\log\lambda_k$ 从而对任意的$\delta>0$

$$(\gamma_k/\lambda_k)(1-\exp(-\lambda_k u))\leqslant\frac{\gamma_k}{\lambda_k}\left(\frac{2\log(\lambda_k\vee e)}{\log(1/u)}\right)^{1+\delta}.$$

因此我们得

$$\begin{aligned}\Lambda^2(u)&=2\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)(1-\exp(-\lambda_k u))\\&\leqslant 2\sum_{k\in K_1}(\gamma_k/\lambda_k)u^{1/2}+2\sum_{k\in K_2}\frac{\gamma_k}{\lambda_k}\left(\frac{2\log(\lambda_k\vee e)}{\log(1/u)}\right)^{1+\delta}\\&\leqslant 2\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)(2\log(\lambda_k\vee e))^{1+\delta}\\&\quad\times(u^{1/2}+(\log(1/u))^{-(1+\delta)}).\end{aligned}$$

由此有

$$(3.3.6)\quad\begin{aligned}&\int_0^{1/e}\frac{\Lambda(u)}{u(\log(1/u))^{1/2}}du\\&\leqslant\left(2\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}(2\log(\lambda_k\vee e))^{1+\delta}\right)^{1/2}\\&\quad\times\int_0^{1/e}\frac{(u^{1/2}+(\log(1/u))^{-(1+\delta)})^{1/2}}{u(\log(1/u))^{1/2}}du\\&=D\left(2\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}(2\log(\lambda_k\vee e))^{1+\delta}\right)^{1/2},\end{aligned}$$

其中$D$是（3.3.6）中第一个不等式右边的那个积分所表示的有限值。

现在我们可以应用引理1.5.1来证明(3.3.4)了。记那个引理中的$a=e$，$\varepsilon=x\left(\Gamma+4\int_0^{1/e}\frac{\Lambda(u)}{u(\log(1/u))^{1/2}}du\right)$，其中$\Gamma^2=$

$EX_{m,n}^2(t)=\sum_{k=n+1}^{m}\gamma_k/\lambda_k$。由引理1.5.1和(3.3.6)式，我们得

$$(3.3.7)\qquad P\{\sup_{0\leq t\leq 1}|X_{mn}(t)|>\varepsilon\}$$

$$\leqslant c\varepsilon^2\left\{1-\Phi\left[\varepsilon/\left(\Gamma+4\int_0^{1/e}A(u)/u(\log(1/u))^{\frac{1}{2}}du\right)\right]\right\}$$

$$\leqslant c\varepsilon^2\left\{1-\Phi\left[\varepsilon/\left(\left(\sum_{k=n+1}^{m}\gamma_k/\lambda_k\right)^{\frac{1}{2}}\right.\right.\right.$$

$$\left.\left.\left.+4D\left(2\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)(2\log(\lambda_k Ve))^{1+\delta}\right)^{\frac{1}{2}}\right)\right]\right\}.$$

利用条件(3.3.2)，当$n\to\infty$时，上式右边趋于0。因此，注意到过程的平稳性，对一切充分大的$n$，我们有

$$P\{\sup_{|t|\leq T}|X_{mn}(t)|>\varepsilon\}$$
$$\leqslant 2(T+1)P\{\sup_{0\leq t\leq 1}|X_{mn}(t)|>\varepsilon\}\leqslant\eta.$$

这就完成了定理的证明。

### 3.2 连续模

为了证明关于连续模的结论，首先建立几个大偏差的不等式。记

$$(3.3.8)\qquad \sigma^2(s)=E(X(t+s)-X(t))^2.$$

**引理3.3.1** 假设满足条件(3.3.2)，又设由(3.3.8)式定义的$\sigma(\cdot)$是一个具有正指数的、在零点正则变化的函数，也就是说

$$(3.3.9)\qquad \sigma(s)=s^{\alpha}L(s),\ \alpha>0,$$

其中$L(\cdot)$是一个在零点缓变的函数，也即它是可测的、正的，且对所有的$\lambda>0$

$$\lim_{s\downarrow 0}L(\lambda s)/L(s)=1.$$

那么，对任意的$\varepsilon>0$，存在常数$C=C(\varepsilon)>0$和$0<h(\varepsilon)<1$，使得对一切$v>0$和$0<h<h(\varepsilon)$，成立着

$$(3.3.10)\ P\{\sup_{0\leq t\leq 1-h}\sup_{0\leq s\leq h}|X(t+s)-X(t)|\geqslant v\sigma(h)\}$$

$$\leqslant \frac{C}{h}\exp\left(-\frac{v^2}{2+\varepsilon}\right).$$

**证** 对任意的正实数 $r$ 和 $t$，记$R=2^r$，$t_r=[2^r t]/2^r$。写

$$(3.3.11)\quad |X(t+s)-X(t)| \leqslant |X((t+s)_r)-X(t_r)| + \sum_{j=0}^{\infty}|X((t+s)_{r+j+1})-X((t+s)_{r+j})| + \sum_{j=0}^{\infty}|X(t_{r+j+1})-X(t_{r+j})|.$$

对$0<h<1$和$u>0$，我们有

$$(3.3.12)\quad P\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}|X((t+s)_r)-X(t_r)| \geqslant u\sigma(h+R^{-1})\} \leqslant 2R(Rh+1)e^{-u^2/2},$$

$$(3.3.13)\quad P\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}|X((t+s)_{r+j+1})-X((t+s)_{r+j})| \geqslant (u^2+2j)^{1/2}\sigma(2^{-(r+j+1)})\} \leqslant 2\exp\left(-\frac{u^2+2j}{2}\right)2^{r+j+1} \leqslant 4Re^{-u^2/2}(2/e)^j$$

和

$$(3.3.14)\quad P\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}|X(t_{r+j+1})-X(t_{r+j})| \geqslant (u^2+2j)^{1/2}\sigma(2^{-r+j+1})\} \leqslant 4Re^{-u^2/2}(2/e)^j.$$

因此

$$(3.3.15)\quad P\left\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}|X(t+s)-X(t)| \geqslant u\sigma(h+R^{-1}) + 2\sum_{j=0}^{\infty}(u^2+2j)^{1/2}\sigma(2^{-(r+j+1)})\right\} \leqslant (2R(Rh+1)+8RD)e^{-u^2/2},$$

其中$D=\sum_{j=0}^{\infty}(2/e)^j$。取$R$使之满足$2R>A/h\geqslant R$，其中的$A$是将在后面指定的正常数。因为$L(\cdot)$是缓变函数，所以对任意给定的$0<\varepsilon<1$，只要取 $r$ 充分大（相当于取 $h$ 充分小），就有

$$\sigma(2^{-(r+j+1)})=2^{-(r+j+1)\alpha}L(2^{-(r+j+1)})$$

$$\leqslant(1+\varepsilon)2^{-(r+j+1)\alpha}L(2^{-(r+j)})$$
$$\leqslant(1+\varepsilon)^{j+1}2^{-(r+j+1)\alpha}L(2^{-r})$$
$$\leqslant 2^{-(j+1)\alpha/2}\left(h\frac{1}{hR}\right)^{\alpha}L\left(h\frac{1}{hR}\right)$$
$$\leqslant 2^{-(j+1)\alpha/2}\left(\frac{2}{A}\right)^{\alpha}\left(1+\frac{\varepsilon}{9}\right)h^{\alpha}L(h)$$
$$=\left(1+\frac{\varepsilon}{9}\right)\left(\frac{2}{A}\right)^{\alpha}2^{-(j+1)\alpha/2}\sigma(h).$$

于是我们得

$$(3.3.16)\quad u\sigma(h+R^{-1})+2\sum_{j=0}^{\infty}(u^2+2j)^{1/2}\sigma(2^{-(r+j+1)})$$
$$\leqslant u\sigma(h)\left(1+\frac{\varepsilon}{9}\right)\left(1+\frac{2}{A}\right)^{\alpha}+2\left(\frac{2}{A}\right)^{\alpha}\left(1+\frac{\varepsilon}{9}\right)$$
$$\times\sum_{j=0}^{\infty}(u^2+2j)^{1/2}2^{-(j+1)\alpha/2}\sigma(h)$$
$$\leqslant u\sigma(h)\left(1+\frac{\varepsilon}{9}\right)\left[\left(1+\frac{2}{A}\right)^{\alpha}+2\left(\frac{2}{A}\right)^{\alpha}\sum_{j=0}^{\infty}2^{-(j+1)\alpha/2}\right]$$
$$+2\left(\frac{2}{A}\right)^{\alpha}\left(1+\frac{\varepsilon}{9}\right)\sigma(h)\sum_{j=0}^{\infty}(2j)^{1/2}/2^{(j+1)\alpha/2}.$$

令$u=v\Big/\left(1+\frac{\varepsilon}{8}\right)$. 因为不失一般性我们总可假设$v\geqslant1$，所以只要$A$取得足够大，(3.16) 右端就不大于

$$u\sigma(h)\left(1+\frac{\varepsilon}{8}\right)=v\sigma(h).$$

此外，

$$(3.3.17)\quad(2R(Rh+1)+8RD)e^{-u^2/2}\leqslant(D'A/h)e^{-v^2/(2+\varepsilon)},$$

其中$D'=2(A+1)+8D$. 将(3.3.16)和(3.3.17)代入(3.3.15)，即得待证的 (3.3.10) 式.

**注3.3.1** 引理的证明没有利用过程的具体分布. 因此，对于任何a.s.连续的Gauss过程，只要其均值为0，(3.3.8)中的$\sigma^2(\cdot)$

满足(3.3.9),那么引理的结论仍成立(参见Csáki,Csörgő,Lin和Révész (1990))。

**注3.3.2** (3.3.10) 可改写成

$$P\{\sup_{|t|\leqslant T}\sup_{0\leqslant s\leqslant h}|X(t+s)-X(t)|\geqslant v\sigma(h)\}$$
$$\leqslant\frac{CT}{h}\exp\left(-\frac{v^2}{2+\varepsilon}\right).$$

**定理3.3.2**(Csáki等, 1990) 假设$\{X(t)\}$满足引理3.3.1的条件,那么成立着

$$(3.3.18)\lim_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}\frac{|X(t+s)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}}=1\qquad \text{a.s.}$$

$$(3.3.19)\lim_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1-h}\frac{|X(t+h)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}}=1\qquad \text{a.s.}$$

**证** 利用引理3.3.1,如Csörgő和Révész (1981) 中Lévy连续模定理的第一部分的证明(参见该书26页),我们能够得到

$$(3.3.20)\quad \overline{\lim_{h\downarrow 0}}\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}\frac{|X(t+s)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}}\leqslant 1\qquad \text{a.s.}$$

因此,为了得到定理的结论,只需证明

$$(3.3.21)\quad \underline{\lim_{h\downarrow 0}}\sup_{0\leqslant t\leqslant 1-h}\frac{|X(t+h)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}}\geqslant 1\qquad \text{a.s.}$$

对$0\leqslant i<j$

$$(3.3.22)E(X((i+1)h)-X(ih))(X((j+1)h)-X(jh))$$
$$=\sum_{k=0}^{\infty}(\gamma_k/\lambda_k)e^{-\lambda_k(j-i)}(2-e-e^{-1})<0.$$

因此,由引理1.1.1我们有

$$(3.3.23)P\left\{\sup_{0\leqslant t\leqslant 1-h}\frac{|X(t+h)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}}\leqslant(1-\varepsilon)^{1/2}\right\}$$
$$\leqslant P\left\{\max_{0\leqslant i\leqslant 1/h-1}\frac{X((i+1)h)-X(ih)}{\sigma(h)}\right.$$
$$\left.\leqslant(2(1-\varepsilon)\log(1/h))^{1/2}\right\}$$
$$\leqslant\left\{1-\frac{h^{1-\varepsilon}}{(16\pi\log(1/h))^{1/2}}\right\}^{(1/h)}$$

$$\leqslant \exp\left\{-\frac{h^{-\varepsilon}}{(16\pi\log(1/h))^{1/2}}\right\}.$$

令$h=h_n=1/n$，则由上面的不等式推得

$$\sum_{n=1}^{\infty} P\left\{\sup_{0\leqslant t\leqslant 1-1/n}\frac{|X(t+1/n)-X(t)|}{\sigma(1/n)(2\log n)^{1/2}}\leqslant(1-\varepsilon)^{1/2}\right\}<\infty.$$

于是从Borel-Cantelli引理得到

$$(3.3.24)\ \varliminf_{n\to\infty}\sup_{0\leqslant t\leqslant 1-1/n}\frac{|X(t+1/n)-X(t)|}{\sigma(1/n)(2\log n)^{1/2}}\geqslant(1-\varepsilon)^{1/2}\quad \text{a.s.}$$

考虑$1/(n+1)<h\leqslant 1/n$。我们有

$$\sup_{0\leqslant t\leqslant 1-h}\frac{|X(t+h)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}}$$

$$\geqslant \sup_{0\leqslant t\leqslant 1-1/n}\frac{|X(t+1/n)-X(t)|}{\sigma(1/n)(2\log n)^{1/2}}\cdot\frac{\sigma(1/n)(2\log n)^{1/2}}{\sigma(h)(2\log(1/h))^{1/2}}$$

$$-2\sup_{0\leqslant t\leqslant 1-\frac{1}{n(n+1)}}\ \sup_{0\leqslant s\leqslant\frac{1}{n(n+1)}}\frac{|X(t+s)-X(t)|}{\sigma\left(\frac{1}{n(n+1)}\right)(2\log(n(n+1)))^{1/2}}$$

$$\times\frac{\sigma\left(\frac{1}{n(n+1)}\right)(2\log(n(n+1)))^{1/2}}{\sigma(h)(2\log(1/h))^{1/2}}.$$

因为$\sigma(\cdot)$在零点是正则变化的，故对$1/(n+1)<h\leqslant 1/n$有

$$(3.3.25)\ \lim_{n\to\infty}\sigma(1/n)(2\log n)^{1/2}/(\sigma(h)(2\log(1/h))^{1/2})=1$$

和

$$(3.3.26)\qquad \lim_{n\to\infty}\sigma\left(\frac{1}{n(n+1)}\right)(2\log(n(n+1)))^{1/2}$$

$$/(\sigma(h)(2\log(1/h))^{1/2})=0.$$

综合（3.3.24），（3.3.20）和（3.3.25），（3.3.26）等结果，由上面给出的不等式即可推出（3.21）。定理证毕。

**注3.3.3** 回顾注3.3.2，我们能够将(3.3.18)和(3.3.19)改写如下：假设$T_h$是当$h\to 0$时不减地趋于无穷的连续函数，则有

(3.3.27) $$\lim_{h\downarrow 0}\sup_{|t|\leqslant T_h}\sup_{0\leqslant s\leqslant h}\frac{|X(t+s)-X(t)|}{\sigma(h)(2\log(T_h/h))^{1/2}}=1 \quad \text{a.s.}$$

(3.3.28) $$\lim_{h\downarrow 0}\sup_{|t|\leqslant T_h}\frac{|X(t+h)-X(t)|}{\sigma(h)(2\log(T_h/h))^{1/2}}=1 \quad \text{a.s.}$$

(Csörgő, Lin 1990b).

### 3.3.3 整体连续模

对于Wiener过程$\{W(t),t\geqslant 0\}$，Csörgő和Révész(1981) 曾指出如下的整体连续模定理：

$$\overline{\lim_{t\to 0}}\frac{|W(t)|}{\sqrt{2t\log\log(1/t)}}=1 \quad \text{a.s.}$$

$$\lim_{t\to 0}\sup_{0\leqslant s\leqslant t}\frac{|W(s)|}{\sqrt{2s\log\log(1/s)}}=1 \quad \text{a.s.}$$

(Csörgő和Révész (1981)定理1.3.3). 但他们所指出的证明方法只适用于前一式，对后者是行不通的，但可直接给出证明.对过程$\{X(t),\ t\geqslant 0\}$也有相应的整体连续模定理.

**定理3.3.3** 设过程$\{X(t),\ t\geqslant 0\}$满足 (3.3.1), (3.3.2)和

(3.3.29) $$\Gamma_1=\sum_{i=1}^{\infty}\gamma_i<\infty.$$

那么我们有

(3.3.30) $$\overline{\lim_{h\to 0}}\sup_{0\leqslant s\leqslant h}\frac{|X(s)-X(0)|}{\sqrt{2\sigma^2(h)\log\log(1/h)}}=1 \quad \text{a.s.}$$

(3.3.31) $$\overline{\lim_{h\to 0}}\frac{|X(h)-X(0)|}{\sqrt{2\sigma^2(h)\log\log(1/h)}}=1 \quad \text{a.s.}$$

**证** 首先来证

(3.3.32) $$\overline{\lim_{h\to 0}}\sup_{0\leqslant s\leqslant h}\frac{|X(s)-X(0)|}{\sigma(h)\sqrt{2\log\log(1/h)}}\leqslant 1 \quad \text{a.s.}$$

记$Y(s)=X(sh)-X(0)$, $0\leqslant s\leqslant 1$, 它是Gauss过程, $EY(s)=0$且

$$(3.3.33)\ E|Y(t)-Y(s)|^2=2\sum_{i=1}^{\infty}\frac{\gamma_i}{\lambda_i}(1-\exp(-h|t-s|\lambda_i))$$

$$\leqslant 2\Gamma_1 h|t-s| =: \Lambda^2(|t-s|),$$

其中 $\Lambda(x)=x\sqrt{2\Gamma_1 h}$ 满足引理1.5.1的条件. 又当$0\leqslant t\leqslant 1$时

$$(3.3.34)\ E|Y^2(t)|=2\sum_{i=0}^{\infty}\frac{\gamma_i}{\lambda_i}(1-\exp(-th\lambda_i))\leqslant 2\Gamma_1 h=:\Gamma^2$$

$$(\Gamma>0).$$

另一方面，由$\Gamma_1<\infty$，直接验证可知

$$\lim_{h\to 0}\sigma^2(h)/2\Gamma_1 h=1.$$

所以对任给$\varepsilon>0$，从$\sigma^2(h)$ 的单调不减性，有$h_0>0$ 使对 $0\leqslant h\leqslant h_0$有

$$(3.3.35)\qquad (1-\varepsilon/2)2\Gamma_1 h\leqslant\sigma^2(h)\leqslant 2\Gamma_1 h.$$

对于给定的$\varepsilon>0$，有充分大的 $a$ 使

$$(3.3.36)\ 4\int_1^{\infty}\Lambda(a^{-u^2})du=4\sqrt{2\Gamma_1 h}\int_1^{\infty}a^{-u^2/2}du\leqslant\varepsilon\sqrt{2\Gamma_1 h}.$$

由 (3.3.33)—(3.3.36)，利用引理1.5.1，我们有

$$P\Big\{\sup_{0\leqslant s\leqslant h}|X(s)-X(0)|/\sigma(h)\sqrt{2\log\log(1/h)}\geqslant(1+\varepsilon)^3\Big\}$$

$$\leqslant P\Big\{\sup_{0\leqslant t\leqslant 1}|Y(t)|\geqslant(1+\varepsilon)\Big(\sqrt{2\Gamma_1 h}$$

$$+4\int_1^{\infty}\Lambda(a^{-u^2})du\Big)\sqrt{2\log\log(1/h)}\Big\}$$

$$\leqslant ca^2\int_{(1+\varepsilon)\sqrt{2\log\log(1/h)}}^{\infty}\exp\Big(-\frac{u^2}{2}\Big)du$$

$$\leqslant ca^2(\log(1/h))^{1+\varepsilon}.$$

令$h_k=\theta^{-k}$，$\theta>1$. 由Borel-Cantelli 引理得

$$\overline{\lim_{k\to\infty}}\sup_{0\leqslant s\leqslant h_k}\frac{|X(s)-X(0)|}{\sigma(h_k)\sqrt{2\log\log(1/h_k)}}\leqslant(1+\varepsilon)^3\quad\text{a.s.}$$

对任给的$h>0$，有 $h$ 使$h_{k+1}\leqslant h<h_k$，注意到

$$\overline{\lim_{h\to 0}}\sup_{0\leqslant s\leqslant h}\frac{|X(s)-X(0)|}{\sigma(h)\sqrt{2\log\log(1/h)}}$$

$$\leqslant\overline{\lim_{k\to\infty}}\sup_{0\leqslant s\leqslant h_k}\frac{|X(s)-X(0)|}{\sigma(h_k)\sqrt{2\log\log(1/h_k)}}\quad\frac{\sigma(h_k)}{\sigma(h_{k+1})}$$

$$\leqslant(1+\varepsilon)^3\sqrt{\theta}.$$

证$\varepsilon\to 0, \theta\to 1$得证 (3.32) 成立.

其次，我们来证

(3.3.37) $\overline{\lim_{h\to 0}}|X(h)-X(0)|/\sqrt{2\sigma^2(h)\log\log(1/h)}\geqslant 1$ a.s.

令$h_n=e^{-n}$，记

$$Y_n=(X(h_n)-X(0))/\sigma(h_n),$$

$$A_n=\{Y_n\geqslant(1-\varepsilon)\sqrt{2\log\log(1/h_n)}\}.$$

直接计算易知

(3.3.38) $$\sum_{n=1}^{\infty}P(A_n)=\infty.$$

为证(3.3.37)成立，只需证$P\{A_n \text{ i.o.}\}=1$. 由引理1.5.4只需验证引理1.5.4的条件 (ii) 被满足. 由 (3.3.35) 知存在$N_0$使当$n\geqslant N_0$时

(3.3.39) $(1-\varepsilon)2\Gamma_1 h_n\leqslant\sigma^2(h_n)\leqslant 2\Gamma_1 h_n$

写

$$I_n=\sum_{1\leqslant j<k\leqslant n}\{P(A_jA_k)-P(A_j)P(A_k)\}$$

$$=\sum_{j=1}^{N_1-1}\sum_{k=j+1}^{n}\{P(A_jA_k)-P(A_j)P(A_k)\}$$

$$+\sum_{k=N_1-1}^{n}\sum_{j=N_1}^{k-1}\{P(A_jA_k)-P(A_j)P(A_k)\}$$

$$=:I_{1n}(N_1)+I_{2n}(N_1).$$

这里$N_1$在下面确定.

设$N_2\geqslant N_0$，记$u_k=\log k$，由引理1.5.3知

$$(3.3.40)\quad |I_{2n}(N_2)| \leqslant \sum_{k=N_2+1}^{n} \sum_{j=N_2}^{k-1} |r_{jk}| \phi(\lambda_j, \lambda_k; r^*_{jk})$$

$$= \left( \sum_{k=N_2+1}^{n} \sum_{j=N_2}^{k-n_k} + \sum_{k=N_2+1}^{n} \sum_{j=k-n_k+1}^{k-1} \right)$$

$$\frac{|r_{jk}|}{2\pi\sqrt{1-r^{*2}_{jk}}} \exp\left( -\frac{\lambda_j^2 + \lambda_k^2 - 2\lambda_j\lambda_k r^*_{jk}}{2(1-r^{*2}_{jk})} \right)$$

$$=: J_{1n}(N_2) + J_{2n}(N_2),$$

其中$r_{jk} = EY_jY_k$，$\lambda_j = (1-\varepsilon)\sqrt{2\log\log(1/h_j)} = (1-\varepsilon) \times \sqrt{2\log j}$，$r^*_{jk}$在 0 与$r_{jk}$之间，由 (3.3.39)，当$j<k$时

$$(3.3.41)\quad 0 < r_{jk} = E(X(h_k)-X(0))(X(h_j)-X(0))/\sigma(h_k)\sigma(h_j)$$

$$\leqslant \sigma(h_k)/\sigma(h_j) \leqslant ((1-\varepsilon)e)^{-1} =: r.$$

所以

$$(3.3.42)\quad J_{1n}(N_2) \leqslant \sum_{k=N_2+1}^{n} \sum_{j=N_2}^{k-n_k} \frac{r_{jk}\lambda_j\lambda_k}{\sqrt{1-r^2}} \psi(\lambda_j)\psi(\lambda_k)$$

$$\times \exp\left\{ -\frac{1}{2(1-r^{*2}_{jk})} (r^*_{jk}(\lambda_j^2 + \lambda_k^2) - 2\lambda_k\lambda_j r^*_{jk}) \right\}$$

$$\leqslant \sum_{k=N_2+1}^{n} \sum_{j=N_2}^{k-n_k} \frac{r_{jk}\lambda_k^2}{\sqrt{1-r^2}} \exp(r_{jk}\lambda_k^2)\psi(\lambda_j)\psi(\lambda_k),$$

其中 $\psi(x) = \dfrac{1}{x\sqrt{2\pi}} \exp\left(-\dfrac{x^2}{2}\right)$. 由(3.3.41)当 $k-j \geqslant n_k = \log k$ 时

$$r_{jk} \leqslant \sqrt{\frac{1}{1-\varepsilon} e^{-(k-j)}} \leqslant \sqrt{1/k(1-\varepsilon)},$$

$$r_{jk}\lambda_k^2 \leqslant (c\log k)/\sqrt{k}.$$

故对任给$\delta>0$，可取$N_3$使当$k>N_3$时有

$$(3.3.43)\qquad r_{jk}\lambda_k^2 \exp(r_{jk}\lambda_k^2) < \delta\sqrt{1-r^2},$$

这样从 (3.3.42)和(3.3.43)，当让$N_1 = N_2 \vee N_3$时就有

(3.3.44) $$J_{1n}(N_1)\leqslant\delta\Big(\sum_{j=1}^{n}P(A_j)\Big)^2.$$

注意到当$k-u_k+1\leqslant j\leqslant k-1$且$j$足够大时有

(3.3.45) $$j+1\leqslant k\leqslant j+2\log j.$$

所以有

(3.3.46) $$J_{7n}(N_2)\leqslant\sum_{j=N_1-u_{N_1}+1}^{n-1}\sum_{k=j+1}^{j+2\log j}\frac{r_{jk}\lambda_j\psi(\lambda_j)}{\sqrt{2\pi}\sqrt{1-r^2}}$$

$$\times\exp\left\{-\frac{(\lambda_k-r_{jk}^*\lambda_j)^2}{2(1-r_{jk}^{*2})}\right\}$$

$$\leqslant\sum_{j=N_1-u_{N_{1+1}}+1}^{n-1}(2\log j)\cdot c\sqrt{2\log j}\,\psi(\lambda_j)$$

$$\times\ j^{-\frac{1-r}{1+r}(1-\varepsilon)^2}$$

$$\leqslant c\sum_{j=1}^{n}P(A_j).$$

将 (3.3.44), (3.3.46) 代入 (3.3.40) 得

$$|I_{2n}(N_1)|\leqslant\delta\Big(\sum_{j=1}^{n}P(A_j)\Big)^2+c\sum_{j=1}^{n}P(A_j).$$

而

$$|I_{1n}(N_1)|\leqslant 2N_1\sum_{j=1}^{n}P(A_j).$$

这就证明了引理1.5.4的条件 (ii) 被满足. (3.3.37)得证.

## § 3.4 $l^2$模平方过程

$l^2$模平方过程$\chi^2(\cdot)$由下式定义

$$\chi^2(t)=\|Y(t)\|^2=\sum_{k=1}^{\infty}X_k^2(t),\quad -\infty<t<\infty.$$

如同我们在§3.1中指出的，它也是一个与无穷维OU过程$\{Y(t),-\infty<t<\infty\}=\{X_k(t),\ -\infty<t<\infty\}_{k=1}^{\infty}$紧密相关的过程。已经有若干学者研究了它的轨道性质。因为$\chi^2(\cdot)$不是一个Gauss过程，所以我们无法应用那些对Gauss过程的研究十分有用的性质（如Fernique引理，Slepian引理）。

附加于$\Gamma_0<\infty$，我们还需要条件$\Gamma_2=\sum_{k=1}^{\infty}\gamma_k^2/\lambda_k<\infty$。

### 3.4.1 连续模

为了建立$\chi^2(\cdot)$的连续模，我们也需要一些大偏差的结果。因为它们是针对非Gauss过程的，因此这些结果也有它们自身的意义。记$M=\max_{j\geq 1}\gamma_j^2/\lambda_j$。

引理3.4.1 假设$\Gamma_0<\infty$且$\Gamma_2<\infty$。那么对任意的$\varepsilon>0$，存在$h(\varepsilon)>0$和$C=C(\varepsilon)>0$，使得对任意的$T>h(\varepsilon)$，$h<h(\varepsilon)$和$t\geq 0$，$v\geq(8/\varepsilon^2)(\Gamma_2/M)^{1/2}$，成立着

$$(3.4.1)\quad P\{|\chi^2(t+h)-\chi^2(t)|\geq v(8hM)^{1/2}\}\geq\frac{1}{7v}\exp\left(-\frac{v}{1-\varepsilon}\right)$$

和

$$(3.4.2)\quad P\{\sup_{|t|\leq T}\ \sup_{0\leq s\leq h}|\chi^2(t+s)-\chi^2(t)|\geq v(8hM)^{1/2}\}$$

$$\leq\frac{CT}{h}\exp\left(-\frac{v}{1+\varepsilon}\right).$$

证 记$M_n=\max_{1\leq j\leq n}\gamma_j^2/\lambda_j$，$\sigma_k^2=E(X_k(t+h)+X_k(t))^2$和$\sigma_k'^2=E(X_k(t+h)-X_k(t))^2$。于是

$$(3.4.3)\qquad E(X_k^2(t+h)-X_k^2(t))^2=\sigma_k^2\sigma_k'^2$$
$$=4(\gamma_k/\lambda_k)^2(1-\exp(-2\lambda_k h)).$$

令

$$p_n(v)=p\left\{\left|\sum_{j=1}^{n}(X_j^2(t+h)-X_j^2(t))\right|\geq v(8hM_n)^{1/2}\right\}.$$

首先我们证明：对充分大的$n$，$v\geqslant(8/\varepsilon^2)(\Gamma_2/M_n)^{1/2}$，

$$(3.4.4)\quad \frac{1}{7v}\exp\left(-\frac{v}{1-\varepsilon}\right)\leqslant p_n(v)\leqslant 2\exp\left(-\frac{v}{1+\varepsilon}\right).$$

令$k_0$是这样一个整数，使得$\gamma_{k_0}^2/\lambda_{k_0}=M_n$。记

$$Y=\sum_{j=1}^{n}(X_j^2(t+h)-X_j^2(t))-(X_{k_0}^2(t+h)-X_{k_0}^2(t)).$$

因此$Y$和$X_{k_0}^2(t+h)-X_{k_0}^2(t)$是相互独立的。又因

$$\sum_{j=1}^{n}(X_j^2(t+h)-X_j^2(t))$$

$$=\sum_{j=1}^{n}(X_j(t+h)+X_j(t))(X_j(t+h)-X_j(t))$$

是对称的，所以

$$(3.4.5)\quad p_n(v)=2P\left\{\sum_{j=1}^{n}(X_j^2(t+h)-X_j^2(t))\geqslant v(8hM_n)^{1/2}\right\}$$

$$\geqslant 2P\{X_{k_0}^2(t+h)-X_{k_0}^2(t)\geqslant v(8hM_n)^{1/2},\ Y\geqslant 0\}$$

$$=2P\{X_{k_0}^2(t+h)-X_{k_0}^2(t)\geqslant v(8hM_n)^{1/2}\}P(Y\geqslant 0)$$

$$\geqslant P\{X_{k_0}^2(t+h)-X_{k_0}^2(t)\geqslant v(8hM_n)^{1/2}\}.$$

我们来估计概率$P\{X_k^2(t+h)-X_k^2(t)\geqslant v\sigma_k\sigma_k'\}$。记$f_k$为随机变量$X_k^2(t+h)-X_k^2(t)$的密度函数。由$X_k(t+h)+X_k(t)$和$X_k(t+h)-X_k(t)$之间的独立性，我们有

$$f_k(x)=\frac{1}{\pi\sigma_k\sigma_k'}\int_0^\infty\frac{1}{y}\exp\left\{-\frac{x^2}{2\sigma_k^2y^2}-\frac{y^2}{2\sigma_k'^2}\right\}dy.$$

由此，并利用正态分布的尾概率估计，对充分大的$v$得到

$$(3.4.6)\quad P\{X_k^2(t+h)-X_k^2(t)\geqslant v\sigma_k\sigma_k'\}$$

$$=\frac{1}{\pi\sigma_k\sigma_k'}\int_0^\infty\frac{1}{y}\left(\int_{v\sigma_k\sigma_k'}^\infty\exp\left\{-\frac{x^2}{2\sigma_k^2y^2}\right\}dx\right)$$

$$\times\exp\left\{-\frac{y^2}{2\sigma_k'^2}\right\}dy$$

$$\geqslant\frac{1}{\pi\sigma_k'^2v}\int_0^\infty y\left(1-\frac{y^2}{v^2\sigma_k'^2}\right)\exp\left\{-\frac{v^2\sigma_k'^2}{2y^2}-\frac{y^2}{2\sigma_k'^2}\right\}dy$$

$$\geqslant \frac{1}{\pi v}\int_{v^{1/2}}^{v^{3/4}} y\left(1-\frac{y^2}{v^2}\right)\exp\left\{-\frac{v^2}{2y^2}-\frac{y^2}{2}\right\}dy$$

$$\geqslant \frac{e^v}{\pi v^{1/2}}\int_{2v^{1/2}}^{v^{3/4}+v^{1/4}} \exp\left(-\frac{x^2}{2}\right)dx$$

$$\geqslant \frac{e^v}{\pi v^{1/2}}\left\{\left(\frac{1}{2v^{1/2}}-\frac{1}{8v^{3/2}}\right)\exp(-2v)\right.$$

$$\left.-\frac{1}{v^{3/4}+v^{1/4}}\exp\{-(v^{3/4}+v^{1/4})^2/2\}\right\}$$

$$\geqslant \frac{1}{7v}\exp(-v).$$

这里我们利用了变量代换$x=y+v/y$.由（3.4.5）和（3.4.6），并注意到当$h\to 0$时

$$\sigma_{k_0}^2\sigma_{k_0}'^2/(8hM_n)\to 1,$$

得证（3.4.4）左边的不等式.

下面我们来证明（3.4.4）的右边部分.对$0\leqslant x\leqslant 1/(\sigma_j\sigma_j')$，我们有

$$\begin{aligned}
&E\exp\{x(X_j^2(t+h)-X_j^2(t))\}\\
&=E\{E[\exp\{x(X_j(t+h)+X_j(t))(X_j(t+h)\\
&\quad -X_j(t))\}\mid X_j(t+h)+X_j(t)]\}\\
&=E\exp\left\{\frac{1}{2}x^2(X_j(t+h)+X_j(t))^2\sigma_j'^2\right\}\\
&=(1-x^2\sigma_j'^2\sigma_j^2)^{-1/2}.
\end{aligned}$$

因此对$0\leqslant x\leqslant 1/(\sigma_{k_0}\sigma_{k_0}')$，成立

$$(3.4.7)\quad p_n(v)\leqslant 2\exp\{-xv(8hM_n)^{-1/2}\}\prod_{j=1}^{n}(1-x^2\sigma_j^2\sigma_j'^2)^{-1/2}.$$

令$x=(1-\varepsilon/2)/(\sigma_{k_0}\sigma_{k_0}')$.因而

$$x^2\sigma_{k_0}^2\sigma_{k_0}'^2\leqslant(1-\varepsilon/2)^2\leqslant 1-3\varepsilon/4.$$

我们将要利用下列不等式：对$0<\varepsilon<1$和$0\leqslant y\leqslant 1-\varepsilon$，

(3.4.8) $$1-y\geqslant e^{-y/\varepsilon}.$$

由此我们得

$$\prod_{j=1}^{n}(1-x^2\sigma_j^2\sigma_j'^2)^{-1/2}\leqslant\exp\left\{\frac{1}{2}\frac{4}{3\varepsilon}x^2\sum_{j=1}^{n}\sigma_j^2\sigma_j'^2\right\}$$

$$\leqslant\exp\left\{\frac{1}{\varepsilon}\sum_{j=1}^{n}\sigma_j^2\sigma_j'^2/(\sigma_{k_0}^2\sigma_{k_0}'^2)\right\}.$$

将它代入(3.4.7),并注意到假设$v\geqslant(8/\varepsilon^2)(\Gamma_2/M_n)^{1/2}$,当$h$足够小时就有

$$p_n(v)\leqslant2\exp\left\{-(1-\varepsilon/2)v(8hM_n)^{1/2}/(\sigma_{k_0}\sigma_{k_0}')\right.$$

$$\left.+\frac{1}{\varepsilon}\sum_{j=1}^{n}\sigma_j^2\sigma_j'^2/(\sigma_{k_0}^2\sigma_{k_0}'^2)\right\}$$

$$\leqslant2\exp\{-(1-2\varepsilon/3)v\}\leqslant2\exp\left(-\frac{v}{1+\varepsilon}\right).$$

这就完成了(3.4.4)的证明.

由假设$\Gamma_2<\infty$可知，对一切充分大的$n$，$M_n=M$.对所有这样的$n$，$p_n(v)$的定义中的$M_n$换作$M$时,(3.4.4)式仍然成立.因此由(3.4.4),对$v\geqslant(8/\varepsilon^2)(\Gamma_2/M)^{1/2}$,

(3.4.9) $$\frac{1}{7v}\exp\left(-\frac{v}{1-\varepsilon}\right)\leqslant P\{|\chi^2(t+h)-\chi^2(t)|$$

$$\geqslant v(8hM)^{1/2}\}\leqslant2\exp\left(-\frac{v}{1+\varepsilon}\right).$$

(3.4.9)的左边的不等式即是 (3.4.1).从(3.4.9) 的右边的不等式,沿着引理3.3.2的证明路线,又可证得(3.4.2).具体步骤从略.

**定理3.4.1** (Csörgő and Lin, 1990b)假设$\Gamma_0<\infty$且$\Gamma_2<\infty$.又设$T_h$是当$h\to0$时不减地趋于无穷的连续函数.那么

$$\lim_{h\downarrow0}\sup_{|t|\leqslant T_h}\sup_{0\leqslant s\leqslant h}\frac{|\chi^2(t+s)-\chi^2(t)|}{(8hM)^{1/2}\log(T_h/h)}\leqslant1\ \text{a.s.}$$

如果$T_h$还满足条件

(3.4.10) $$(\log T_h)/\log(1/h)\to\infty,\quad h\to0,$$

那么

$$\lim_{h\downarrow 0}\sup_{|t|\leqslant T_h}\sup_{0\leqslant s\leqslant h}\frac{|\chi^2(t+s)-\chi^2(t)|}{(8hM)^{1/2}\log(T_h/h)}=1\quad \text{a.s.}$$

$$\lim_{h\downarrow 0}\sup_{|t|\leqslant T_h}\frac{|\chi^2(t+h)-\chi^2(t)|}{(8hM)^{1/2}\log(T_h/h)}=1\qquad \text{a.s.}$$

**证** 对给定的$0<\varepsilon<1$，令$h_n$满足

$$\sum_{n=1}^{\infty}(h_{n-1}/T_{h_n})^{\varepsilon/2}<\infty, \tag{3.4.11}$$

且当$n\to\infty$时

$$(h_{n-1}/T_{h_n})/(h_n/T_{h_{n+1}})\to 1. \tag{3.4.12}$$

由引理3.4.1，我们有

$$\begin{aligned}P\{&\sup_{|t|\leqslant T_{h_n}}\sup_{0\leqslant s\leqslant h_{n-1}}|\chi^2(t+s)-\chi^2(t)|\\&\geqslant(1+2\varepsilon)(8h_{n-1}M)^{1/2}\log(T_{h_n}/h_{n-1})\}\\&\leqslant(CT_{h_n}/h_{n-1})\exp\left\{-\frac{1+2\varepsilon}{1+\varepsilon}\log(T_{h_n}/h_{n-1})\right\}\\&\leqslant C(h_{n-1}/T_{h_n})^{\varepsilon/2}.\end{aligned}$$

将它与（3.4.11）相结合得到

$$\varlimsup_{n\to\infty}\sup_{|t|\leqslant T_{h_n}}\sup_{0\leqslant s\leqslant h_{n-1}}\frac{|\chi^2(t+s)-\chi^2(t)|}{(8h_{n-1}M)^{1/2}\log(T_{h_n}/h_{n-1})}\leqslant 1+2\varepsilon\qquad \text{a.s.}$$

由此，再利用（3.4.12）产生

$$\varlimsup_{h\downarrow 0}\sup_{|t|\leqslant T_h}\sup_{0\leqslant s\leqslant h}\frac{|\chi^2(t+s)-\chi^2(t)|}{(8hM)^{1/2}\log(T_h/h)}\leqslant 1\qquad \text{a.s.} \tag{3.4.13}$$

为了完成定理的证明，只需验证：在条件（3.4.10）下，成立着

$$\varliminf_{h\downarrow 0}\sup_{|t|\leqslant T_h}\frac{|\chi^2(t+s)-\chi^2(t)|}{(8hM)^{1/2}\log(T_h/h)}\geqslant 1-\varepsilon\qquad \text{a.s.} \tag{3.4.14}$$

由条件$\Gamma_2<\infty$，类似于（4.13），只要$K=K$（$\varepsilon$）充分大，易知

$$(3.4.15)\quad \overline{\lim_{h\downarrow 0}}\ \sup_{|t|\le T_h} \frac{\left|\sum_{k=K}^{\infty}(X_k^*(t+h)-X_k^*(t))\right|}{(8hM)^{1/2}\log(T_h/h)}\le\varepsilon\quad\text{a.s.}$$

固定$K$充分大，由（3.4.15），（3.4.14）等价于

$$(3.4.16)\quad \lim_{h\downarrow 0}\ \sup_{|t|\le T_h} \frac{\left|\sum_{k=1}^{K}(X_k^*(t+h)-X_k^*(t))\right|}{(8hM)^{1/2}\log(T_h/h)}\ge 1-\varepsilon\quad\text{a.s.}$$

定义$h_n$使得 $T_{h_{n-1}}/h_n=n$. 记$\xi_l^k=X_k((l+1)h_n)-X_k(lh_n)$, $\eta_l^k=X_k((l+1)h_n)+X_k(lh_n)$.则有

$$\sigma_{l\,l}^k:=E(\xi_l^k)^2=\frac{2\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_n}),$$

$$\alpha_{l\,l}^k:=E(\eta_l^k)^2=\frac{2\gamma_k}{\lambda_k}(1+e^{-\lambda_k h_n}),$$

又对$l<r$,

$$\sigma_{l\,r}^k:=E\xi_l^k\xi_r^k=\frac{\gamma_k}{\lambda_k}e^{-\lambda_k(r-l)h_n}(2-e^{-\lambda_k h_n}-e^{\lambda_k h_n}),$$

$$\alpha_{l\,r}^k:=E\eta_l^k\eta_r^k=\frac{\gamma_k}{\lambda_k}e^{-\lambda_k(r-l)h_n}(2+e^{-\lambda_k h_n}+e^{\lambda_k h_n}),$$

$$\tau_{l\,r}^k:=E\xi_l^k\eta_r^k=\frac{\gamma_k}{\lambda_k}e^{-\lambda_k(r-l)h_n}(e^{\lambda_k h_n}-e^{-\lambda_k h_n})$$

（显然，$\tau_{r\,l}^k=-\tau_{l\,r}^k$）.令

$$\xi_{1,l}^k=\xi_l^k-\frac{\sigma_{1l}^k}{\sigma_{11}^k}\xi_1^k-\frac{\tau_{l1}^k}{\alpha_{11}^k}\eta_1^k,\quad \eta_{1,l}^k=\eta_l^k-\frac{\tau_{1l}^k}{\sigma_{11}^k}\xi_1^k-\frac{\alpha_{1l}^k}{\alpha_{11}^k}\eta_1^k.$$

容易看出（$\xi_1^k,\eta_1^k$）和（$\xi_{1,l}^k,\eta_{1,l}^k$）是相互独立的.写

$$\sum_{k=1}^{K}\xi_l^k\eta_l^k=\sum_{k=1}^{K}\Big\{\xi_{1,l}^k\eta_{1,l}^k+\frac{\sigma_{1l}^k}{\sigma_{11}^k}\xi_1^k\eta_l^k-\frac{\sigma_{1l}^k\tau_{1l}^k}{(\sigma_{11}^k)^2}(\xi_1^k)^2$$

$$-\frac{\sigma_{1l}^k\alpha_{1l}^k}{\sigma_{11}^k\alpha_{11}^k}\xi_1^k\eta_1^k+\frac{\tau_{l1}^k}{\alpha_{11}^k}\eta_1^k\eta_l^k-\frac{\tau_{1l}^k}{\sigma_{11}^k}\ \frac{\tau_{l1}^k}{\alpha_{11}^k}\xi_1^k\eta_1^k-\frac{\alpha_{1l}^k\tau_{l1}^k}{(\alpha_{11}^k)^2}(\eta_1^k)^2$$

$$+\frac{\tau_{1l}^k}{\sigma_{11}^k}\xi_1^k\xi_l^k+\frac{\alpha_{1l}^k}{\alpha_{11}^k}\xi_l^k\xi_1^k\Big\}$$

$$=\sum_{k=1}^{K}\xi_{1,l}^{k}\eta_{1,l}^{k}+H_{l}.$$

又记

$$\lambda^{m}=\min\{\lambda_k, k\leqslant K\}>0,$$
$$L=[(\lambda^{m}h_n)^{-1}\log(T_{h_{n-1}}/h_n)],$$
$$A_n=(1-\varepsilon)(8h_nM)^{1/2}\log(T_{h_{n-1}}/h_n).$$

我们有

$$(3.4.17)\quad P\Big\{\max_{|l|\leqslant[T_{h_{n-1}}/h_n]}\Big|\sum_{k=1}^{K}(X_k^k((l+1)h_n)-X_k^k(lh_n))\Big|\leqslant A_n\Big\}$$
$$\leqslant P\Big\{\max_{1\leqslant j\leqslant T_{h_{n-1}}/(Lh_n)}\Big|\sum_{k=1}^{K}(X_k^k((jL+1)h_n)-X_k^k(jLh_n))\Big|\leqslant A_n\Big\}$$
$$\leqslant P\Big\{\Big|\sum_{k=1}^{K}(X_k^k((L+1)h_n)-X_k^k(Lh_n))\Big|\leqslant A_n\Big\}$$
$$\times P\Big\{\max_{2\leqslant j\leqslant T_{h_{n-1}}/(Lh_n)}\Big|\sum_{k=1}^{K}\xi_{1,jL}^{k}\eta_{1,jL}^{k}\Big|\leqslant A_n(1+(T_{h_{n-1}}/h_n)^{-1})\Big\}$$
$$+P\{\max_{1\leqslant j\leqslant T_{h_{n-1}}/(Lh_n)}|H_{jL}|>A_n(T_{h_{n-1}}/h_n)^{-1}\}$$

我们先来估计上式中最后的那个概率。考虑 $H_{jL}$ 中代表性的一项 $\sum_{k=1}^{K}(\sigma_{1,jL}^{k}/\sigma_{11}^{k})\xi_1^k\eta_{jL}^{k}$。记 $\bar{\eta}_{jL}^{k}=\eta_{jL}^{k}-(\tau_{1,jL}^{k}/\sigma_{11}^{k})\xi_1^k$。它与 $\xi_1^k$ 是独立的，且有

$$E(\bar{\eta}_{jL}^{k})^2=\sigma_{jL,jL}^{k}-(\tau_{1,jL}^{k})^2/\sigma_{11}^{k}.$$

易知估计 $\sum_{k=1}^{K}(\sigma_{1,jL}^{k}/\sigma_{11}^{k})\xi_1^k\eta_{jL}^{k}$ 的概率可以代之以估计 $\sum_{k=1}^{K}(\sigma_{1,jL}^{k}/\sigma_{11}^{k})\times\xi_1^k\bar{\eta}_{jL}^{k}$ 的相应的概率. 模仿(3.4.4)的右边的不等式的证明, 可得

$$(3.4.18)\ P\left\{\left|\sum_{k=1}^{K}(\sigma_{1,jL}^{k}/\sigma_{11}^{k})\xi_{1}^{k}\bar{\eta}_{jL}^{k}\right|>\frac{1}{16}A_n(T_{h_{n-1}}/h_n)^{-1}\right\}$$

$$\leqslant 2\exp\left\{-\frac{1}{16}\left(1-\frac{\varepsilon}{2}\right)A_n(T_{h_{n-1}}/h_n)^{-1}/\right.$$

$$((\sigma_{1,jL}^{k_0})^2(\sigma_{11}^{k_0})^{-1}(\alpha_{jL,jL}^{k}-(\tau_{1,jL}^{k_0})^2/\sigma_{11}^{k_0}))^{1/2}$$

$$+\frac{1}{\varepsilon}\left(\sum_{k=1}^{K}(\sigma_{1,jL}^{k})^2(\sigma_{11}^{k})^{-1}(\alpha_{jL,jL}^{k}-(\tau_{1,jL}^{k})^2/\sigma_{11}^{K})^2(\sigma_{1,jL}^{k_0})^2\right.$$

$$\left.\left.\times(\sigma_{11}^{k_0})^{-1}(\alpha_{jL,jL}^{k_0}-(\tau_{1,jL}^{k_0})^2/\sigma_{11}^{k_0})\right)\right\},$$

其中

$$(\sigma_{1,jL}^{k})^2(\sigma_{11}^{k})^{-1}\leqslant(\sigma_{1L}^{k})^2(\sigma_{11}^{k})^{-1}$$
$$=O((\gamma_k\lambda_k h_n^2(T_{h_{n-1}}/h_n)^{-1})^2/(\gamma_k/\lambda_k))$$
$$=O(\gamma_k\lambda_k^3 h_n^6 T_{h_{n-1}}^{-2}),$$
$$\alpha_{jL,jL}^{k}-(\tau_{1,jL}^{k})^2/\sigma_{11}^{k}\sim 2\gamma_k/\lambda_k.$$

将它们代入（3.4.18），对充分大的$n$得到

$$P\left\{\left|\sum_{k=1}^{K}(\sigma_{1,jL}^{k}/\sigma_{11}^{k})\xi_{1}^{k}\bar{\eta}_{jL}^{k}\right|\geqslant\frac{1}{16}A_n(T_{h_{n-1}}/h_n)^{-1}\right\}$$
$$\leqslant 2\exp\{-ch_n^{-3/2}\log(T_{h_{n-1}}/h_n)\}\leqslant(T_{h_{n-1}}/h_n)^{-4}.$$

因此我们就有

$$P\left\{\max_{1\leqslant j\leqslant T_{h_{n-1}}/(Lh_n)}\left|\sum_{k=1}^{K}(\sigma_{1,jL}^{k}/\sigma_{11}^{k})\xi_{1}^{k}\bar{\eta}_{jL}^{k}\right|\right.$$
$$\left.\geqslant\frac{1}{16}A_n(T_{h_{n-1}}/h_n)^{-1}\right\}$$
$$\leqslant L^{-1}(T_{h_{n-1}}/h_n)^{-3}.$$

对于$H_{jL}$中的其它项，也有类似的估计.这样就证明了

$$P\{\max_{1\leqslant j\leqslant T_{h_{n-1}}/(LH_n)}|H_{jL}|\geqslant A_n(T_{h_{n-1}}/h_n)^{-1}\}$$
$$\leqslant cL^{-1}(T_{h_{n-1}}/h_n)^{-3}.$$

类似的程序也适用于估计（3.4.17）右端的第二个概率，得出

$$(3.4.19)\qquad P\left\{\max_{1\leqslant j\leqslant T_{h_{n-1}}/(Lh_n)}\left|\sum_{k=1}^{K}\xi_{1,jL}^{k}\eta_{1,jL}^{k}\right|\right.$$
$$\left.\leqslant A_n(1+(T_{h_{n-1}}/h_n)^{-1})\right\}$$
$$\leqslant P\left\{\max_{1\leqslant j\leqslant T_{h_{n-1}}/(Lh_n)}\left|\sum_{k=1}^{K}\xi_{jL}^{k}\eta_{jL}^{k}\right|\leqslant A_n(1+2(T_{h_{n-1}}/h_n)^{-1})\right\}$$
$$+cL^{-1}(T_{h_{n-1}}/h_n)^{-3}.$$

将这两个估计代入(3.4.17)，并且对(3.4.19)式右边的概率重复同样的手续。继续这一过程，最后可得

$$(3.4.20)\ P\left\{\max_{|l|\leqslant[T_{h_{n-1}}/h_n]}\left|\sum_{k=1}^{K}(X_k^2((l+1)h_n)-X_k^2(lh_n))\right|\right.$$
$$\left.\leqslant A_n\right\}$$
$$\leqslant\prod_{j=1}^{[T_{h_{n-1}}/(Lh_n)]}P\left\{\left|\sum_{k=1}^{K}(X_k^2((jL+1)h_n)-X_k^2(jLh_n))\right|\right.$$
$$\left.\leqslant A_n(1+j(T_{h_{n-1}}/h_n)^{-1})\right\}+c(LT_{h_{n-1}}/h_n)^{-2}$$
$$\leqslant\left(P\left\{\left|\sum_{k=1}^{K}(X_k^2(h_n)-X_k^2(0))\right|\right.\right.$$
$$\left.\left.\leqslant\left(1+\frac{\varepsilon}{3}\right)A_n\right\}\right)^{[T_{h_{n-1}}/(Lh_n)]}+c(LT_{h_{n-1}}/h_n)^{-2}.$$

取$K=K(\varepsilon)$充分大，同时利用(3.4.1)和条件(3.4.10)，对充分大的$n$，(3.4.20)式的右端的第一项不超过

$$P\{|\chi^2(h_n)-\chi^2(0)|$$
$$\leqslant\left(1-\frac{2\varepsilon}{3}\right)(8h_nM)^{1/2}\log(T_{h_{n-1}}/h_n)\})^{[T_{h_{n-1}}/(Lh_n)]}$$

$$\leqslant\left(1-\frac{1}{8\log(T_{h_{n-1}}/h_n)}\exp\left\{-\left(1-\frac{\varepsilon}{2}\right)\right.\right.$$
$$\left.\left.\times\log(T_{h_{n-1}}/h_n)\right\}\right)^{[T_{h_{n-1}}/(Lh_n)]}$$

$$\leqslant \exp\{-(T_{h_{n-1}}/h_n)^{4/3}L^{-1}\}\leqslant \exp\{-T_{h_{n-1}}^{4/4}\}\leqslant n^{-2}.$$

对(3.4.16)的证明的其余部分与(3.3.23)式的有关证明类似，从略。定理证毕。

### 3.4.2 对数型定律

首先，我们给出下列大偏差的结果。记$m=\max\limits_{j\geqslant 1}\gamma_j/\lambda_j$。

**引理3.4.2** 假设$\Gamma_0<\infty$。那么对任意的$\varepsilon>0$，$t\geqslant 0$ 和$v>0$，成立着

$$(3.4.21)\qquad P\{\chi^2(t)\geqslant 2mv\}\geqslant \frac{\varepsilon}{6}v^{1/2}\exp\left(-\frac{v}{1-\varepsilon}\right).$$

如果还满足条件$\Gamma_2<\infty$，那么存在$C=C(\varepsilon)>0$和$v_0=v_0(\varepsilon)>0$，使得对任意的$T>0$和$v\geqslant v_0$，成立着

$$(3.4.22)\qquad P\{\sup_{|t|\leqslant T}\chi^2(t)\geqslant 2mv\}\leqslant CT\exp\left(-\frac{v}{1+\varepsilon}\right).$$

**证** (3.4.22)是Iscoe和McDonald (1989)的定理1的一个推论。因此我们只须证明 (3.4.21)。

记$m_n=\max\limits_{1\leqslant j\leqslant n}\gamma_j/\lambda_j$，$\tau_n=\sum\limits_{j=1}^{n}\gamma_j/\lambda_j$。首先我们证明：对任意的$v>0$，

$$(3.4.23)\qquad P\left\{\sum_{j=1}^{n}X_j^2(t)\geqslant 2m_nv\right\}\geqslant \frac{\varepsilon}{6}v^{1/2}\exp\left(-\frac{v}{1-\varepsilon}\right).$$

记$k_0$是使得$\gamma_{k_0}/\lambda_{k_0}=m_n$的正整数。又记 $Y=\sum\limits_{j=1}^{n}X_j^2(t)-X_{k_0}^2(t)$。它与$X_{k_0}^2(t)$是独立的。由中心极限定理，存在$n_0$，使得对$n\geqslant n_0$，$P\{Y\geqslant EY\}\geqslant\frac{1}{3}$。因此，仿照(3.4.5)，再注意到$X_{k_0}^2(t)/m_n$服从$\chi_1^2$分布，得到

$$(3.4.24)\quad P\left\{\sum_{j=1}^{n}X_j^2(t)\geqslant 2m_nv\right\}\geqslant \frac{1}{3}P\{X_{k_0}^2(t)\geqslant 2m_nv\}$$

$$=\frac{1}{3}\int_{2v}^{\infty}\frac{1}{2^{1/2}\Gamma(1/2)}y^{1/2}e^{-y/2}dy$$

$$\geqslant\frac{1}{3\cdot 2^{1/2}\Gamma(1/2)}\int_{2v}^{2v/(1-\varepsilon)}y^{1/2}e^{-y/2}dy$$

$$\geqslant\frac{\varepsilon}{6}v^{1/2}\exp\left(-\frac{v}{1-\varepsilon}\right).$$

(3.4.23)得证.因为$\Gamma_0<\infty$，由(3.4.23)易知(3.4.21)成立.

**定理3.4.2** (Lin 1990) 假设$\Gamma_0<\infty$且$\Gamma_2<\infty$.那么

$$\lim_{T\to\infty}\sup_{|t|\leqslant T}\frac{\chi^2(t)}{2m\log T}=1 \quad \text{a.s.} \tag{3.4.25}$$

$$\overline{\lim_{T\to\infty}}\frac{\chi^2(T)}{2m\log T}=1 \quad \text{a.s.} \tag{3.4.26}$$

**证** 利用(3.4.22)，容易证明

$$\overline{\lim_{T\to\infty}}\sup_{|t|\leqslant T}\frac{\chi^2(t)}{2m\log T}\leqslant 1 \quad \text{a.s.} \tag{3.4.27}$$

因此，为了证明(3.4.25)，只需对任意的$\varepsilon>0$，验证

$$\underline{\lim}_{T\to\infty}\sup_{|t|\leqslant T}\frac{\chi^2(t)}{2m\log T}\geqslant 1-\varepsilon \quad \text{a.s.} \tag{3.4.28}$$

由$\Gamma_0<\infty$易知$\lim\limits_{k\to\infty}\max\limits_{j\geqslant k}\gamma_j/\lambda_j=0$.因此，类似于(3.4.27)，可以证明：存在正整数$K=K(\varepsilon)$，使得

$$\overline{\lim_{T\to\infty}}\sup_{|t|\leqslant T}\sum_{k=K}^{\infty}X_k^2(t)/2m\log T\leqslant\varepsilon \quad \text{a.s.} \tag{3.4.29}$$

固定这样的值$K$，由(3.4.29)，(3.4.28)等价于

$$\underline{\lim}_{T\to\infty}\sup_{|t|\leqslant T}\sum_{k=1}^{K}X_k^2(t)/2m\log T\geqslant 1-\varepsilon \quad \text{a.s.} \tag{3.4.30}$$

对$1\leqslant i\leqslant j$，记$\chi_{ji}=\sum\limits_{k=1}^{K}X_k^2(j)e^{-2\lambda_k(j-i)}$，$\xi_{ji}=\sum\limits_{k=1}^{K}X_k(j)(X_k(i)-X_k(j)e^{-\lambda_k(j-i)})e^{-\lambda_k(j-i)}$.又记$n'=[n^{1-\varepsilon/6}]$，$n_i=n-i[n^{\varepsilon/6}]$，$i=1$，

$\cdots, n', n_0 = n, \lambda''(K) = \min\limits_{1\leqslant k\leqslant K} \lambda_k$. 对 $i\leqslant n_j$ 和 $k\leqslant K$，我们有(3.4.31)

$$EX_k^2(n_{j-1})e^{-2\lambda_k(n_{j-1}-i)} = \frac{\gamma_k}{\lambda_k}e^{-2\lambda_k(n_{j-1}-i)} \leqslant \frac{\gamma_k}{\lambda_k}e^{-\lambda''(K)n^{\varepsilon/6}}.$$

此外，因为 $\{X_k(j), k=1,\cdots,K\}$ 与 $\{X_k(i)-X_k(j)e^{-\lambda_k(j-i)}, k=1, \cdots, K, i=1,\cdots,j-1\}$ 是独立的，所以 $\sum\limits_{k=1}^{K} X_k^2(j)$ 与 $\sum\limits_{k=1}^{K} X_k^2(i) - \chi_{ji} - 2\xi_{ji}(i<j)$ 也是独立的。因此

$$P\left\{\sup_{|t|\leqslant n}\sum_{k=1}^{K} X_k^2(t)/(2m\log n)\leqslant 1-\varepsilon\right\}$$

$$\leqslant P\left\{\max_{i\leqslant n}\sum_{k=1}^{K} X_k^2(i)/(2m\log n)\leqslant 1-\varepsilon\right\}$$

$$\leqslant P\left\{\sum_{k=1}^{K} X_k^2(n)/(2m\log n)\leqslant 1-\varepsilon\right\}P\left\{\max_{i\leqslant n_1}\left(\sum_{k=1}^{K} X_k^2(i)\right.\right.$$

$$\left.\left.-\chi_{ni}-2\xi_{ni}\right)/(2m\log n)\leqslant 1-\varepsilon+\frac{\varepsilon}{4n}\right\}$$

$$+P\left\{\max_{i\leqslant n}\ \xi_{ni}/(2m\log n)\geqslant\frac{\varepsilon}{8n}\right\}$$

$$\leqslant P\left\{\sum_{k=1}^{K} X_k^2(n)/(2m\log n)\leqslant 1-\varepsilon\right\}P\left\{\max_{i\leqslant n_1}\sum_{k=1}^{K} X_k^2(i)/\right.$$

$$\left.(2m\log n)\leqslant 1-\varepsilon+\frac{\varepsilon}{2n}\right\}+P\left\{\max_{i\leqslant n}\chi_{ni}/(2m\log n)\geqslant\frac{\varepsilon}{12n}\right\}$$

$$+2P\left\{\max_{i\leqslant n}\xi_{ni}/(2m\log n)\geqslant\frac{\varepsilon}{12n}\right\}.$$

归纳地有

(3.4.32) $$P\left\{\sup_{|t|\leqslant n}\sum_{k=1}^{K} X_k^2(t)/(2m\log n)\leqslant 1-\varepsilon\right\}$$

$$\leqslant\prod_{j=1} P\left\{\sum_{k=1}^{K} X_k^2(n_j)/(2m\log n)\leqslant 1-\frac{\varepsilon}{2}\right\}$$

$$+\sum_{j=1}^{n'} P\left\{\max_{i\leqslant n_j} \chi_{n_{j-1}} i/(2m\log n) > \frac{\varepsilon}{12n}\right\}$$

$$+2\sum_{j=1}^{n'} P\left\{\max_{i\leqslant n_j} \xi n_{j-1} i/(2m\log n) > \frac{\varepsilon}{12n}\right\}$$

$$=: p_{n1}+p_{n2}+p_{n3}.$$

在(3.4.21)中分别用 $\frac{\varepsilon}{6}$和$\sum_{k=1}^{K} X_k^2(t)$代替$\varepsilon$和$\chi^2(t)$(回顾(3.4.29)式)，对充分大的$n$就可得到

$$p_{n1}\leqslant\left\{1-\frac{\varepsilon}{54}\log^{\frac{1}{2}} n\exp\left(-\frac{(1-\varepsilon/2)\log n}{1-\varepsilon/6}\right)\right\}^{n'}$$
$$\leqslant\exp(-n^{\varepsilon/6}).$$

对 $p_{n2}$，利用（3.4.22)，同时注意到(3.4.31)，当$x$不很小（例如$x\geqslant n^{-1}$)时，就有

$$P\{\max_{i\leqslant n_j}\chi_{n_{j-1}}\ i\geqslant 2mx\}\leqslant cn_j\exp\left\{-\frac{x}{1+\varepsilon}e^{\lambda''(K)n^{\varepsilon/6}}\right\}$$
$$\leqslant cn\exp\left\{-\frac{x}{1+\varepsilon}e^{\lambda''(K)n^{\varepsilon/6}}\right\}.$$

因此

$$p_{n2}\leqslant cn^{2-\varepsilon/6}\exp\left\{-\frac{\varepsilon\log n}{12(1+\varepsilon)n}e^{\lambda''(K)n^{\varepsilon/6}}\right\}.$$

再来考虑$p_{n3}$.注意到$X_k(j)$与$(X_k(i)-X_k(j)e^{-\lambda_k(j-i)})e^{-\lambda_k(j-i)}$是相互独立的，故对$0\leqslant a\leqslant(\lambda_k/\gamma_k)^{-1}e^{\lambda_k(j-i)}(1-e^{-2\lambda_k(j-i)})^{-1/2}$，有

$$\begin{aligned}
&E\exp\{aX_k(j)(X_k(i)-X_k(j)e^{-\lambda_k(j-i)})e^{-\lambda_k(j-i)}\}\\
&=E\{E[\exp\{aX_k(j)(X_k(i)-X_k(j)e^{-\lambda_k(j-i)})\\
&\quad\times e^{-\lambda_k(j-i)}\}|X_k(j)]\}\\
&=E\exp\left\{\frac{1}{2}a^2X_k^2(j)(\gamma_k/\lambda_k)(1-e^{-2\lambda_k(j-i)})e^{-2\lambda_k(j-i)}\right\}\\
&=(1-a^2(\gamma_k/\lambda_k)^2(1-e^{-2\lambda_k(j-i)})e^{-2\lambda_k(j-i)})^{-1/2}.
\end{aligned}$$

令 $a=\left(1-\frac{\varepsilon}{2}\right)^{1/2}m^{-1}e^{\lambda''(K)(j-i)}$. 利用不等式(3.4.8)，对$j-i\geqslant$

$n^{\varepsilon/6}$有

$$E\exp(a\xi_{ji})=\prod_{k=1}^{K}(1-a^2(\gamma_k/\lambda_k)^2(1-e^{-2\lambda_k(j-i)})e^{-2\lambda_k(j-i)})^{-1/2}$$

$$\leqslant\exp\left\{\frac{1}{2}\frac{2}{\varepsilon}a^2\sum_{k=1}^{K}(\gamma_k/\lambda_k)^2(1-e^{-2\lambda_k(j-i)})e^{-2\lambda_k(j-i)}\right\}$$

$$\leqslant\exp(\Gamma_3/\varepsilon m^2),$$

其中因$\Gamma_0<\infty$，$\Gamma_3=\sum_{k=1}^{\infty}\gamma_k^2/\lambda_k^2<\infty$。 因此对$j-i\geqslant n^{\varepsilon/6}$和$x\geqslant n^{-1}$当$n$充分大时有

$$P\{\xi_{ji}>2mx\}\leqslant\exp\{-2amx+\Gamma_3/\varepsilon m^2\}$$

$$\leqslant\exp\{-xe^{\lambda''(K)n^{\varepsilon/6}}\}.$$

于是

$$p_{n3}\leqslant 2\sum_{j=1}^{n'}n_j\exp\left\{-\frac{\varepsilon\log n}{12n}e^{\lambda''(K)n^{\varepsilon/6}}\right\}$$

$$\leqslant 2n^{2-\varepsilon/6}\exp\left\{-\frac{\varepsilon\log n}{12n}e^{\lambda''(K)n^{\varepsilon/6}}\right\}.$$

结合对$p_{n1}$,$p_{n2}$和$p_{n3}$的估计，从（3.4.32）得到

$$P\left\{\sup_{|t|\leqslant n}\sum_{k=1}^{K}X_k^2(t)/(2m\log n)\leqslant 1-\varepsilon\right\}=O(\exp(-n^{\varepsilon/6})),\ n\to\infty.$$

由此即得

$$(3.4.33)\ \varliminf_{n\to\infty}\sup_{|t|\leqslant n}\sum_{k=1}^{K}X_k^2(t)/(2m\log(n+1))\geqslant 1-\varepsilon\quad \text{a.s.}$$

这样就证明了(3.4.30)，也就是证得(3.4.28)。因此(3.4.25)得证。

（3.4.26）可以沿着从（3.3.29）推出（3.3.30）的路线由(3.4.25)得到。细节从略。

## § 3.5 具有核的两参数Gauss过程

在第3.2节中所研究的两参数Gauss过程$X(t,n)$ 可改写为

$$(3.5.1)\quad X(t,n)=\sum_{k=1}^{n}X_k(t)$$
$$=\sum_{k=1}^{n}\int_{-\infty}^{t}\exp(-\lambda_k(t-s))(2\gamma_k)^{1/2}dW_k(s).$$

由此就引导我们来研究两参数Gauss过程

$$(3.5.2)\qquad X(t,v)=\int_0^v\int_{-\infty}^{t}\exp(-\lambda(y)(t-x))$$
$$\times(2\gamma(y))^{1/2}dW(x,y),$$

其中$\gamma(y)$ 与$\lambda(y)$ 是$[0,\infty)$ 上的正连续函数，$\{W(x,y);\ -\infty<x<\infty,\ 0\leqslant y<\infty\}$ 是标准的两参数Wiener过程（见Csörgő，Lin 1990a）。进一步我们来研究如下形式的Gauss过程$\{X(t,v);\ t\in\boldsymbol{R},\ v\in\boldsymbol{R}^+\}$：

$$(3.5.3)\qquad X(t,v)=\int_0^{\infty}\int_{-\infty}^{\infty}\Gamma(t,v,x,y)dW(x,y),$$

其中核函数$\Gamma(t,v,x,y)$ 是$\boldsymbol{R}^+\times\boldsymbol{R}$ 上关于$(x,y)$ 平方可积的函数.显然$X(t,v)$ 是一个Gauss过程，均值为0，协方差函数为

$$(3.5.4)\qquad \mathrm{Cov}(X(t,v),\ X(s,u))$$
$$=\int_0^{\infty}\int_{-\infty}^{\infty}\Gamma(t,v,x,y)\Gamma(s,u,x,y)dxdy.$$

记

$$H_1^2(t,s,v)=E(X(t+s,v)-X(t,v))^2,$$
$$X(R(t,s,v,u))=X(t+s,v+u)-X(t,v+u)$$
$$-X(t+s,v)+X(t,v),$$
$$H_2^2(t,s,v,u)=E(X(R(t,s,v,u)))^2.$$

容易看出

$$(3.5.5)\quad H_1^2(t,s,v)=\int_0^\infty\int_{-\infty}^\infty(\Gamma(t+s,v,x,y)-\Gamma(t,v,x,y))^2dxdy,$$

$$(3.5.6)\quad H_2^2(t,s,v,u)=\int_0^\infty\int_{-\infty}^\infty(\Gamma(t+s,v+u,x,y)-\Gamma(t,v+u,x,y)-\Gamma(t+s,v,x,y)+\Gamma(t,v,x,y))^2dxdy.$$

下述例子中的过程是熟知的.

例1　若$\Gamma(t,v,x,y)=I_{(-\infty,t]\times[0,v]}(x,y)$，$-\infty<t<\infty$，$0\leqslant v<\infty$，那么

$$X(t,v)=W(t,v),$$

$$H_1^2(t,s,v)=sv,\qquad 0\leqslant s<\infty,$$

$$H_2^2(t,s,v,u)=su,\qquad 0\leqslant s,u<\infty.$$

例2　若$\Gamma(t,v,x,y)=I_{[0,t]\times[0,v]}(x,y)-tI_{[0,1]\times[0,v]}(x,y)$，$0\leqslant t\leqslant 1, 0\leqslant v<\infty$，那么

$$X(t,v)=W(t,v)-tW(1,v)$$

是Kiefer过程（参见Csörgő和Révész（1981）§1.15）.

$$H_1^2(t,s,v)=s(1-s)v,\ \ 0\leqslant s\leqslant 1,$$

$$H_2^2(t,s,v,u)=s(1-s)u,\ \ 0\leqslant s\leqslant 1,\ \ 0\leqslant u<\infty.$$

例3　若$-\infty<t<\infty$，$0<v<\infty$，

$$\Gamma(t,v,x,y)=I_{(-\infty,t]\times(0,v]}(x,y)\exp(-\lambda(y)(t-x))(2\gamma(y))^{1/2},$$

其中$\lambda(y)$和$\gamma(y)$是$(0,\infty)$上正连续函数，那么$X(t,v)$是(3.5.2)中的两参数Gauss过程，具有

$$H_1^2(t,s,v)=2\int_0^v\frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)s))dx,$$

$$H_2^2(t,s,v,u)=2\int_v^{v+u}\frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)s))dx.$$

在本节中，我们将对(3.5.3)定义的过程$X(t,v)$建立连续

模．为简单计，仅仅某些特殊情形被考察了．特别地，我们假设$X(t,v)$ 关于$t$是平稳的．因此我们可写

$$H_1(s,v)=H_1(t,s,v),\quad H_2(s,v,u)=H_2(t,s,v,u).$$

关于连续模和大增量的一般结论可在Csörgő，Lin和Shao(1991)中找到．

此外，在本节中我们总设$H_1(s,v)$关于$s$是不减的，$H_2(s,v,u)$关于$s$和$u$是不减的，$H_1(s,v)$ 和$H_2(s,v,u)$ 对每一变量都是连续的．

### 3.5.1 大偏差

下述引理是Slepian引理的一个推广．

**引理3.5.1** (Gordon，1985)． 设$\{X_{ij}\}_I$，$\{Y_{ij}\}_I$，$I=\{(i,j):1\leqslant i\leqslant n,\ 1\leqslant j\leqslant m\}$是中心化Gauss变量的两个集，满足下列条件：

$$EX_{ij}^2=EY_{ij}^2\qquad (i,j)\in I,$$

$$EX_{ij}X_{ik}\leqslant EX_{ij}Y_{ik},\qquad (i,j),\ (i,k)\in I,$$

$$EX_{ij}X_{lk}\geqslant EY_{ij}Y_{lk},\qquad (i,j),\ (l,k)\in I,\ i\neq l.$$

那么对所有$\lambda_{ij}>0$

$$P\left\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(X_{ij}>\lambda_{ij})\right\} \tag{3.5.7}$$

$$\geqslant P\left\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(Y_{ij}>\lambda_{ij})\right\}.$$

**引理3.5.2** 设$A\subset \boldsymbol{R}^+$，$s_0>0$．假设对任一$t$，$s$和$v\geqslant u$

$$E(X(t+s,v)-X(t,v))(X(t+s,u)-X(t,u)) \tag{3.5.8}$$

$$\geqslant E(X(t+s,u)-X(t,u))^2,$$

且存在$c_0>0$和$\alpha>0$使对任一$T\in A$，$0\leqslant S\leqslant S_1\leqslant S_0$有

$$H_1(S,T)/S^\alpha\leqslant c_0H_1(S_1,T)/S_1^\alpha. \tag{3.5.9}$$

那么对任何$0<\varepsilon<1$存在仅依赖于$c_0$，$\alpha$，和$\varepsilon$的 $C=C(\varepsilon)>0$ 使对任一$x\geqslant 1$有

$$(3.5.10)\ P\left\{\sup_{T\in A}\sup_{|s|\leqslant 1}\sup_{0\leqslant t\leqslant s_0}\frac{|X(t+s,T)-X(t,T)|}{xH_1(s_0,T^*)}\geqslant 1+\varepsilon\right\}$$

$$\leqslant Cs_0^{-1}\exp(-x^2/2),$$

其中$T^*=\sup\{T:T\in A\}$.

**证** 设$Z(T)$是一个独立增量过程，$Z(T)\overset{\mathscr{D}}{=}X(t+s,T)-X(t,T)$.那么$EZ^2(T)=H_1^2(s,T)$且对$T\geqslant T'$由(3.5.8)

$$EZ(T)Z(T')=H_1^2(s,T')$$

$$\leqslant E(X(t+s,T)-X(t,T))(X(t+s,T')-X(t,T')).$$

进一步(3.5.8)还蕴含$H_1(s,T)$是$T$的不减函数.那么由Slepian引理，我们有

$$(3.5.11)\ P\left\{\sup_{T\in A}\frac{|X(t+s,T)-X(t,T)|}{xH_1(s,T^*)}\geqslant 1\right\}$$

$$\leqslant P\left\{\sup_{T\in A}\frac{X(t+s,T)-X(t,T)}{xH_1(s,T^*)}\geqslant 1\right\}$$

$$+P\left\{\sup_{T\in A}\frac{-(X(t+s,T)-X(t,T))}{xH_1(s,T^*)}\geqslant 1\right\}$$

$$\leqslant P\left\{\sup_{T\in A}\frac{Z(T)}{xH_1(s,T^*)}\geqslant 1\right\}+P\left\{\sup_{T\in A}\frac{-Z(T)}{xH_1(s,T^*)}\geqslant 1\right\}$$

$$\leqslant 2P\left\{\sup_{T\in A}\frac{|Z(T)|}{xH_1(s,T^*)}\geqslant 1\right\}\leqslant 4\exp(-x^2/2).$$

记$t_{k+j}=([t2^{2^{k+j}}/s_0]+1)s_0/2^{2^{k+j}}$，$j=0,1,\cdots$.由条件(3.5.9)，对任一$\varepsilon>0$，存在$M>0$使得

$$\int_M^\infty H_1(s_0e^{-z^2},T)dz<\varepsilon.$$

所以，按Fernique的一个结果(参见Jain和Marcus(1978)系2.5)可知，对每一固定的$T$，$X(t,T)$关于$t$是a.s.连续的.因此我们可写

$$(3.5.12)\ |X(t+s,T)-X(t,T)|\leqslant|X((t+s)_k,T)-X(t_k,T)|$$

$$+\sum_{j=0}^{\infty}|X((t+s)_{k+j+1},T)-X((t+s)_{k+j},T)|$$

$$+\sum_{j=0}^{\infty}|X(t_{k+j+1},T)-X(t_{k+j},T)|.$$

令 $K=2^{2^k}$.由$H_1(s,T)$ 的定义及条件(3.5.9)易见对充分大 的 $k$ 和$0\leqslant s\leqslant s_0$有

(3.5.13) $$H_1((t+s)_k-t_k,T)\leqslant H_1(s_0,T)+2H_1(s_0/K,T)$$
$$\leqslant(1+2c_0K^{-\alpha})H_1(s_0,T)\leqslant(1+\varepsilon/2)H_1(s_0,T),$$

(3.5.14) $$H_1((t+s)_{k+j}-(t+s)_{k+j+1},T)\leqslant 2H_1(s_0/2^{2^{k+j}},T)$$
$$\leqslant 2c_0 2^{-\alpha 2^{k+j}}H_1(s_0,T),$$

还有

(3.5.15) $$H_1(t_{k+j}-t_{k+j+1},T)\leqslant 2c_0 2^{-\alpha 2^{k+j}}H_1(s_0,T).$$

从 (3.5.13) 和 (3.5.11)，我们得

(3.5.16) $$P\left\{\sup_{T\in A}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\frac{|X((t+s)_k,T)-X(t_k,T)|}{(1+\varepsilon/2)xH_1(s_0,T^*)}\geqslant 1\right\}$$
$$\leqslant c2^{2^{k+1}}s_0^{-1}\exp(-x^2/2).$$

类似地，由(3.5.14)，(3.5.15)和(3.5.11)，对任一$x_j>0$ 我们有

(3.5.17) $$P\left\{\sup_{T\in A}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\frac{|X((t+s)_{k+j+1},T)-X((t+s)_{k+j},T)|}{2c_0x_j2^{-\alpha 2^{k+j}}H_1(s_0,T^*)}\geqslant 1\right\}$$
$$\leqslant c2^{2^{k+j+1}}s_0^{-1}\exp(-x_j^2/2).$$

和

(3.5.18) $$P\left\{\sup_{T\in A}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\frac{|X(t_{k+j+1},T)-X(t_{k+j},T)|}{2c_0x_j2^{-\alpha 2^{k+j}}H_1(s_0,T^*)}\geqslant 1\right\}$$
$$\leqslant c2^{2^{k+j+1}}s_0^{-1}\exp(-x_j^2/2).$$

从 (3.5.12)，(3.5.16)—(3.5.18) 我们推得

$$(3.5.19)\quad P\Big\{\sup_{T\in A}\sup_{|t|\le 1}\sup_{0\le s\le u_0}|X(t+s,T)-X(t,T)|\Big/\Big(\Big(\Big((1+\varepsilon/2)x+4c_0\sum_{j=0}^{\infty}x_j 2^{-\alpha 2^{k+j}}\Big)H_1(s_0,T^*)\Big)\ge 1\Big\}$$

$$\le cs_0^{-1}\Big\{2^{2^{k+1}}\exp\Big(-\frac{x^2}{2}\Big)+\sum_{j=0}^{\infty}2^{2^{k+j+1}}\exp\Big(-\frac{x_j^2}{2}\Big)\Big\}.$$

令$x_j^2=x^2+2^{k+j+2}$。当$k$充分大时有

$$(3.5.20)\quad \sum_{j=0}^{\infty}2^{2^{k+j+1}}e^{-x_j^2/2}=e^{-x^2/2}\sum_{j=0}^{\infty}(2/e)^{2^{k+j+1}}\le 2e^{-x^2/2},$$

$$(3.5.21)\quad 4c_0\sum_{j=0}^{\infty}x_j 2^{-\alpha 2^{k+j}}\le 4c_0 x\sum_{j=0}^{\infty}2^{-\alpha 2^{k+j}}+4c_0\sum_{j=0}^{\infty}2^{(k+j+2)/2-\alpha 2^{k+j}}<\varepsilon x/2.$$

现在结合（3.5.19），（3.5.20）和（3.5.21）得证（3.5.10）。

**引理3.5.3** 设$A\subset \boldsymbol{R}^+$，$s_0$，$u_0>0$。假设对任何$t,s,0<v'\le v\le b$有

$$(3.5.22)\quad EX(R(t,s,v',b-v'))X(R(t,s,v,b-v))\ge E(X(R(t,s,v,b-v)))^2$$

且存在$c_0>0$和$\alpha>0$使得对$0\le s\le s_1\le s_0$，$0\le v\le 1+u_0$，$0\le u\le 2u_0$有

$$(3.5.23)\qquad H_2(s,v,u)/s^{\alpha}\le c_0H_2(s_1,v,u)/s_1^{\alpha}.$$

此外，假设对任何$\varepsilon>0$存在$\delta_0>0$使对$0<\delta\le\delta_0$有

$$(3.5.24)\quad \sup_{0\le s\le s_0}\sup_{0\le v\le 1+u_0}\sup_{-\delta u_0\le u\le u_0}(H_2(s_0,v+u,\delta u_0)+H_2(\delta s_0,v+u,u_0))/H_2(s_0,v,u_0)\le\varepsilon.$$

那么对任给$0<\varepsilon<1$存在仅依赖于$c_0,\alpha,\varepsilon$的$C=C(\varepsilon)>0$，使得

$$(3.5.25)\quad P\Big\{\sup_{|t|\le 1}\sup_{0\le v\le 1}\sup_{0\le s\le s_0}\sup_{\le u\le u_0}\frac{|X(R(t,s,v,u))|}{xH_2(s_0,v,u_0)}\ge 1\Big\}$$

$$\leqslant Cs_0^{-1}u_0^{-1}\exp(-x^2/2)$$

对任一$x\geqslant 1$成立.

**证** 定义

$$t_{k+j}=([t2^{2^{k+j}}/s_0]+1)s_0/2^{2^{k+j}},$$
$$v'_{k+j}=([v2^{2^{k+j}}/u_0]+1)u_0/2^{2^{k+j}}.$$

注意到 $X(R(t,s,v,u))$被定义于矩形$[t,t+s]\times[v,v+u]$上., 我们有

$$\begin{aligned}(3.5.26)\quad &|X(R(t,s,v,u))|\\ &\leqslant|X(R(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k))|\\ &+\sum_{j=0}^{\infty}|X(R((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},\\ &\quad v'_k,(v+u)'_k-v'_k))|\\ &+\sum_{j=0}^{\infty}|X(R(t_{k+j+1},t_{k+j}-t_{k+j+1},\\ &\quad v'_k,(v+u)'_k-v'_k))|\\ &+|X(R(t,s,v,v'_k-v))|\\ &+|X(R(t,s,v+u,(v+u)'_k-(v+u)))|.\end{aligned}$$

从 (3.5.22) 即得对任何$v\leqslant v'$, $v+u\geqslant v'+u'$有

$$(3.5.27)\qquad H_2(s,v',u')\leqslant H_2(s,v,u).$$

利用 (3.5.24) 和 (3.5.27), 我们得对充分大的$K$

$$\begin{aligned}&H_2((t+s)_k-t_k,v'_k,\ (v+u)'_k-v'_k)\\ &\leqslant H_2(s,v,u)+H_2((t+s)_k-(t+s),\ v'_k,(v+u)'_k-v'_k)\\ &\quad+H_2(t_k-t,\ v'_k,\ (v+u)'_k-v'_k)+H_2(s,v,v'_k-v)\\ &\quad+H_2(s,v+u,(v+u)'_k-(v+u))\\ &\leqslant H_2(s,v,u)+2H_2(s_0/K,v'_k,u_0(1+1/K))\\ &\quad+H_2(s,v,u_0/K)+H_2(s,v+u,u_0/K)\\ &\leqslant(1+\varepsilon/4)H_2(s,v,u).\end{aligned}$$

利用 (3.5.23), (3.5.24) 和 (3.5.27), 类似于 (3.5.14) 和 (3.5.15)我们得

$$H_2((t+s)_{k+j}-(t+s)_{k+j+1},v_k',(v+u)_k'-v_k')$$
$$\leqslant 2H_2(s_0/2^{2^{k+j}},v,u_0(1+1/K))$$
$$\leqslant 2c_0 2^{-\alpha 2^{k+j}} H_2(s_0,v,u_0(1+1/K))$$
$$\leqslant 3c_0 2^{-\alpha 2^{k+j}} H_2(s_0,v,u_0),$$
$$H_2(t_{k+j}-t_{k+j+1},v_k',(v+u)_k'-v_k')$$
$$\leqslant 3c_0 2^{-\alpha 2^{k+j}} H_2(s_0,v,u_0).$$
于是，对任何$x_j>0$，

$$(3.5.28)\quad P\Big\{\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\sup_{0\leqslant u\leqslant u_0}|X(R(t_k,(t+s)_k-t_k,v_k',(v+u)_k'-v_k'))|/((1+\varepsilon/4)xH_2(s,v,u))\geqslant 1\Big\}$$
$$\leqslant 4.2^{2^{k+2}}s_0^{-1}u_0^{-1}\exp(-x^2/2),$$
$$(3.5.29)\quad P\Big\{\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\sup_{0\leqslant u\leqslant u_0}|X(R((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},v_k',(v+u)_k'-v_k'))|/(3c_0x_j2^{-\alpha 2^{k+j}}H_2(s_0,v,u_0))\geqslant 1\Big\}$$
$$\leqslant 4\cdot 2^{2^{k+1}+2^{k+j+1}}s_0^{-1}u_0^{-1}\exp(-x_j^2/2),$$
$$(3.5.30)\quad P\Big\{\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\sup_{0\leqslant u\leqslant u_0}|X(R(t_{k+j+1},t_{k+j}-t_{k+j+1},v_k',(v+u)_k'-v_k'))|/(3c_0x_j2^{-\alpha 2^{k+j}}H_2(s_0,v,u_0))\geqslant 1\Big\}$$
$$\leqslant 4\cdot 2^{2^{k+1}+2^{k+j+1}}s_0^{-1}u_0^{-1}\exp(-x_j^2/2).$$
令$x_j^2=x^2+2^{k+j+2}$.类似于（3.5.20）和（3.5.21），有

$$(3.5.31)\qquad \sum_{j=0}^{\infty}2^{2^{k+1}+2^{k+j+1}}e^{-x_j^2/2}\leqslant 2K^2e^{-x^2/2},$$

$$(3.5.32)\qquad 6c_0\sum_{j=0}^{\infty}x_j2^{-\alpha 2^{k+j}}\leqslant \varepsilon x/4.$$

现在来处理(3.5.26)的最后第二项．令$d_i=(i+1)u_0/K$．对

任一$y>0$我们有

$$(3.5.33)\ P\{\sup_{|t|\leq 1}\sup_{0\leq v\leq 1}\sup_{0\leq s\leq s_0}|X(R(t,\ s,\ v,\ v_k'-v))|\geq y\}$$

$$\leq P\{\max_{0\leq i\leq K/u_0}\sup_{d_{i-1}\leq v\leq d_i}\sup_{|t|\leq 1}\sup_{0\leq s\leq s_0}|X(R(t,\ s,\ v,\ v_k'-v))|\geq y\}$$

$$\leq \sum_{i=0}^{[K/u_0]} P\{\sup_{d_{i-1}\leq v\leq d_i}\sup_{|t|\leq 1}\sup_{0\leq s\leq s_0}|X(R(t,s,v,d_i-v))|\geq y\}.$$

设$Z(\cdot)$是一个独立增量过程，$Z(d_i-v)\overset{\mathscr{D}}{=}X(R(t,\ s,\ v,d_i-v))$，其中$d_{i-1}\leq v<d_i$.那么对任何$v\geq v'$

$$EZ(d_i-v)Z(d_i-v')=EZ^2(d_i-v)$$
$$=EX^2(R(t,\ s,\ v,\ d_i-v))$$
$$\leq EX(R(t,\ s,\ v,\ d_i-v))X(R(t,\ s,\ v',\ d_i-v')),$$

最后一个不等式是由 (5.22) 得出的.由 (5.27),我们得对任一$d_{i-1}\leq v<d_i$有

$$H_2(s,\ v-u_0/K,\ 2u_0/K)\geq H_2(s,\ d_{i-1},\ u_0/K).$$

所以，利用Slepian引理，我们得

$$(3.5.34)\ P\left\{\sup_{d_{i-1}\leq v<d_i}\frac{|X(R(t,s,v,d_i-v))|}{xH_2(s,v-u_0/K,2u_0/K)}\geq 1\right\}$$

$$\leq 2P\left\{\sup_{d_{i-1}\leq v<d_i}\frac{|Z(d_i-v)|}{xH_2(s,v-u_0/K,2u_0/K)}\geq 1\right\}$$

$$\leq 2P\left\{\frac{|Z(d_i-d_{i-1})|}{xH_2(s,d_{i-1},u_0/K)}\geq 1\right\}\leq 4\exp(-x^2/2).$$

又注意到 (3.5.24)，(3.5.34) 蕴含着

$$(3.5.35)\ P\left\{\sup_{d_{i-1}\leq v<d_i}\frac{|X(R(t,s,v,d_i-v))|}{(\varepsilon/16)xH_2(s,v,u_0)}\geq 1\right\}$$

$\leqslant 4\exp(-x^2/2)$.

沿着（3.5.10）的证明思路，我们从（3.5.35）可推得

$$(3.5.36)\quad P\left\{\sup_{d_{i-1}\leqslant v<d_i}\ \sup_{|t|\leqslant 1}\ \sup_{0\leqslant s\leqslant s_0}\frac{|X(R(t,s,v,d_i-v))|}{(\varepsilon/8)xH_2(s,v,u_0)}\geqslant 1\right\}$$

$$\leqslant 8\cdot 2^{2^{k+1}}s_0^{-1}\exp(-x^2/2).$$

综合（3.5.33）和（3.5.36）得到

$$(3.5.37)\quad P\left\{\sup_{|t|\leqslant 1}\ \sup_{0\leqslant v\leqslant 1}\ \sup_{0\leqslant s\leqslant s_0}\frac{|X(R(t,s,v,v_k'-v))|}{(\varepsilon/8)xH_2(s,v,u_0)}\geqslant 1\right\}$$

$$\leqslant 16\cdot 2^{2^{k+2}}s_0^{-1}u_0^{-1}\exp(-x^2/2).$$

类似地，对（3.5.26）的最后一项，我们有

$$(3.5.38)\quad P\left\{\sup_{|t|\leqslant 1}\ \sup_{0\leqslant v\leqslant 1}\ \sup_{0\leqslant s\leqslant s_0}|X(R(t,s,v+u,(v+u)_k-(v\right.$$

$$\left.+u)))|/((\varepsilon/8)xH_2(s,v,u_0))\geqslant 1\right\}$$

$$\leqslant 16\cdot 2^{2^{k+2}}s_0^{-1}u_0^{-1}\exp(-x^2/2).$$

现在，从（3.5.26），(3.5.28)—(3.5.32)，(3.5.37)和(3.5.38)即得(3.5.25)。引理3.5.3证毕。

**注 3.5.1** 从引理3.5.2和3.5.3的证明易见，"$\sup\limits_{|t|\leqslant 1}$"和"$\sup\limits_{0\leqslant v\leqslant 1}$"可改写为"$\sup\limits_{|t|\leqslant M}$"和"$\sup\limits_{0\leqslant v\leqslant M}$"，其中 M>0 是常数．此时，在（3.5.10）或（3.5.25）中，$C=C(\varepsilon)$应用$C=C(\varepsilon,\ M)$代替。

### 3.5.2 轨道性质

应用$X(t,\ v)$的大偏差结果，我们来建立它的连续模．设$a_T$和$b_T$是非负趋于0的连续函数。

**定理 3.5.1**(Csörgő, Lin and Shao, 1991)假设条件(3.5.8)和(3.5.9)被满足．又假设

$$(3.5.39)\quad E(X((i+1)s,\ v)-X(is,\ v))(X((j+1)s,\ u)$$

$$-X(js,u))\leqslant 0,$$

(3.5.40) $\log\log(H_1(a_T,\ T)+H_1^{-1}(a_T,\ T))$
$$=o(\log(1/a_T))\quad T\to\infty.$$

那么我们有

(3.5.41) $$\lim_{T\to\infty}\sup_{|t|\leqslant 1}\frac{|X(t+a_T,T)-X(t,\ T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}}=1\ \text{a.s.}$$

(3.5.42) $$\lim_{T\to\infty}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\frac{|X(t+s,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}}=1\ \text{a.s.}$$

**证** 首先，我们来证

(3.5.43) $$\overline{\lim_{T\to\infty}}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\frac{|X(t+s,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}}$$
$$\leqslant 1\quad \text{a.s.}$$

设$\theta>1$.定义$A_{kj}=\{T:\ \theta^{-(j+1)}<a_T\leqslant\theta^{-j},\ \theta^k\leqslant H_1(\theta^{-j},T)\leqslant\theta^{k+1}\}$, $j=0,\ 1,\ \cdots,\ k=\cdots,\ -1,\ 0,\ 1,\ \cdots,\ T_{kj}^*=\sup\{T:\ T\in A_{kj}\}$, $T'_{kj}=inf\{T:\ T\in A_{kj}\}$.从条件(3.5.40)，对给定的$0<\varepsilon<1/2$和充分大的$j$,当$|k|\geqslant\theta^{\varepsilon j}$时，$A_{kj}=\varnothing$.利用这一事实并注意到条件(3.5.9)且取$\theta$充分接近于1，我们有

(3.5.44) $$\overline{\lim_{T\to\infty}}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\frac{|X(t+s,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}}$$
$$\leqslant\overline{\lim_{j\to\infty}}\max_{|k|\leqslant\theta^{\varepsilon j}}\sup_{T\in A_{kj}}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant a_T}|X(t+s,T)-X(t,$$
$$T)|/(H_1(a_T,T)(2\log(1/a_T))^{1/2})$$
$$\leqslant\overline{\lim_{j\to\infty}}\max_{|k|\leqslant\theta^{\varepsilon j}}\sup_{T\in A_{kj}}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant\theta^{-j}}(1+\varepsilon)|X(t+s,T)$$
$$-X(t,T)|/(\theta^k(2\log\theta^j)^{1/2})$$
$$\leqslant\overline{\lim_{j\to\infty}}\max_{|k|\leqslant\theta^{\varepsilon j}}\sup_{T\in A_{kj}}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant\theta^{-j}}\theta(1+\varepsilon)|X(t+s,T)$$
$$-X(t,T)|/(H_1(\theta^{-j},T_{kj}^*)(2\log\theta^j)^{1/2}.)$$

由引理3.5.2和(3.5.9)，我们得

$$(3.5.45)\ P\Big\{\max_{|k|\le\theta^{\varepsilon j}}\sup_{T\in A_{kj}}\sup_{|t|\le 1}\sup_{0\le s\le\theta^{-j}}|X(t+s,T)-X(t,T)|$$

$$/(H_1(\theta^{-j},T^*_{kj})(2\log\theta^j)^{1/2})\geqslant 1+\varepsilon\Big\}$$

$$\leqslant\sum_{|k|\le\theta^{\varepsilon j}}P\Big\{\sup_{T\in A_{kj}}\sup_{|t|\le 1}\sup_{0\le s\le\theta^{-j}}|X(t+s,\ T)-X(t,$$

$$T)|/(H_1(\theta^{-j},T^*_{kj})(2\log\theta^j)^{1/2})\geqslant 1+\varepsilon\Big\}$$

$$\leqslant C\theta^{\varepsilon j+j}\exp\{-(1+\varepsilon)^2\log\theta^j\}\leqslant C\theta^{-\varepsilon j}.$$

从(3.5.44),(3.5.45)及Borel-Cantelli引理即得(3.5.43)式.

其次，我们来证

$$(3.5.46)\quad \varliminf_{T\to\infty}\sup_{|t|\le 1}\frac{|X(t+a_T,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}}\geqslant 1\ \text{a.s.}$$

它同(3.5.43)一起可推得定理3.5.1的结论.我们有

$$(3.5.47)\varliminf_{T\to\infty}\sup_{|t|\le 1}\frac{|X(t+a_T,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}}$$

$$\geqslant\varliminf_{j\to\infty}\min_{|k|\le\theta^{\varepsilon j}}\inf_{T\in A_{kj}}\sup_{0\le t\le 1}\frac{|X(t+a_T,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}}$$

$$\geqslant\varliminf_{j\to\infty}\min_{|k|\le\theta^{\varepsilon j}}\inf_{T\in A_{kj}}\sup_{0\le t\le 1}\frac{|X(t+\theta^{-j},T)-X(t,T)|}{H_1(\theta^{-j},\ T)(2\log\theta^{j+1})^{1/2}}$$

$$-\varlimsup_{j\to\infty}\max_{|k|\le\theta^{\varepsilon j}}\sup_{T\in A_{kj}}\sup_{0\le t\le 1}\frac{|X(t+\theta^{-j},T)-X(t,T)|}{H_1(\theta^{-j},T)(2\log\theta^j)^{1/2}}$$

$$\geqslant\varliminf_{j\to\infty}\min_{|k|\le\theta^{\varepsilon j}}\inf_{T\in A_{kj}}\max_{0\le i\le\theta^j}|X\ ((i+1)\ \theta^{-j},\ T)$$

$$-X(i\theta^{-j},T)|/(H_1(\theta^{-j},\ T)(2\log\theta^{j+1})^{1/2})$$

$$-\varlimsup_{j\to\infty}\max_{|k|\le\theta^{\varepsilon j}}\max_{T\in A_{kj}}\sup_{a_T\le t\le a_T+1}\sup_{0\le s\le(\theta-1)\theta^{-j-1}}|X(t$$

$$+s,T)-X(t,T)|/(H_1(\theta^{-j},T)(2\log\theta^j)^{1/2}).$$

沿着(3.5.43)的证明方法并注意到注3.5.1，由(3.5.9)我们有

$$(3.5.48)\ \overline{\lim_{j\to\infty}}\ \max_{|k|\leqslant\theta^{\varepsilon j}}\ \sup_{T\in A_{kj}}\ \sup_{a_T\leqslant t\leqslant a_T+1}\ \sup_{0\leqslant s\leqslant(\theta-1)\theta^{-j-1}}|X(t+s,T)-X(t,T)|/(H_1(\theta^{-j},T)(2\log\theta^j)^{1/2})$$

$$\leqslant\overline{\lim_{j\to\infty}}\ \max_{|k|\leqslant\theta^{\varepsilon j}}\ \sup_{T\in A_{kj}}\ \sup_{a_T\leqslant t\leqslant a_T+1}\ \sup_{0\leqslant s\leqslant(\theta-1)\theta^{-j-1}}\varepsilon|X(t+s,T)-X(t,T)|/(H_1((\theta-1)\theta^{-j-1},T)(2\log\theta^j)^{1/2})$$

$$\leqslant\varepsilon \qquad \text{a.s.}$$

设$Y(i, T)=X((i+1)\theta^{-j}, T)-X(i\theta^{-j}, T)$，$Z(i, T)$ 是一个具独立增量的Gauss过程，对每一固定的$i$，$Z(i, T)\stackrel{\mathscr{D}}{=}Y(i, T)$ 且

$$EZ(i, T)Z(j, T')=EY(i, T)Y(j, T') \quad 当 i\neq j.$$

那么由(3.5.8)我们有

$$EY(i,T)Y(i, T')\geqslant EY^2(i, T\wedge T')$$
$$=EZ(i,T)Z(i, T').$$

因此，我们可以利用引理3.5.1得到

$$(3.5.49)\ P\left\{\inf_{T\in A_{kj}}\ \max_{0\leqslant i\leqslant\theta^j}\frac{Y(i, T)}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}}\leqslant\frac{1}{(1+\varepsilon)^2}\right\}$$

$$=1-P\left\{\bigcap_{T\in A_{kj}}\ \bigcup_{0\leqslant i\leqslant\theta^j}\left(\frac{Y(i, T)}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}}\geqslant\frac{1}{(1+\varepsilon)^2}\right)\right\}$$

$$\leqslant 1-P\left\{\bigcap_{T\in A_{kj}}\ \bigcup_{0\leqslant i\leqslant\theta^j}\left(\frac{Z(i,T)}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}}\geqslant\frac{1}{(1+\varepsilon)^2}\right)\right\}$$

$$=P\left\{\bigcup_{T\in A_{kj}}\ \bigcap_{0\leqslant i\leqslant\theta^j}\left(\frac{Z(i,T)}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}}\right.\right.$$

$$\leqslant \frac{1}{(1+\varepsilon)^2}\Big)\Big\}.$$

所以

$$(3.5.50)\quad P\Big\{\min_{|k|\leqslant\theta^{\varepsilon j}}\ \inf_{T\in A_{kj}}\ \max_{0\leqslant i\leqslant\theta^j}\ \frac{Y(i,\ T)}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}}$$

$$\leqslant \frac{1}{(1+\varepsilon)^2}\Big\}$$

$$\leqslant \sum_{|k|\leqslant\theta^{\varepsilon j}} P\Big\{\max_{0\leqslant i\leqslant\theta^j}\frac{Z(i,\ T^*_{kj})}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}\leqslant\frac{\theta}{1+\varepsilon}\Big\}$$

$$+\sum_{|k|\leqslant\theta^{\varepsilon j}} P\Big\{\max_{0\leqslant i\leqslant\theta^j}\ \sup_{T\in A_{kj}}\ \frac{|Z(i,T^*_{kj})-Z(i,T)|}{H_1(\theta^{-j},\ T^*_{kj})(2\log\theta^{j+1})^{1/2}}$$

$$\geqslant\frac{\theta\varepsilon}{(1+\varepsilon)^2}\Big\}.$$

注意到对于固定的$i$，$Z(i,\ T)$ 是一个独立增量过程，我们有

$$\begin{aligned}
&E(Z(i,\ T^*_{kj})-Z(i,\ T'_{kj}))^2\\
&=EZ^2(i,\ T^*_{k})-EZ^2(i,\ T'_{kj})\\
&=EY^2(i,T^*_{kj})-EY^2(i,T'_{kj})\\
&\leqslant\theta^{2(k+1)}-\theta^{2k}\\
&\leqslant(\theta^2-1)H_1^2(\theta^{-j},T^*_{kj}).
\end{aligned}$$

因此对$1<\theta<1+\varepsilon^2/32$

$$(3.5.51)\quad \sum_{|k|\leqslant\theta^{\varepsilon j}} P\Big\{\max_{0\leqslant i\leqslant\theta^j}\ \sup_{T\in A_{kj}}\ \frac{|Z(i,T^*_{kj})-Z(i,T)|}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}$$

$$\geqslant\frac{\theta\varepsilon}{(1+\varepsilon)^2}\Big\}$$

$$\leqslant\sum_{|k|\leqslant\theta^{\varepsilon j}}\ \sum_{i=0}^{\theta^j} P\Big\{\sup_{T\in A_{kj}}\ \frac{|Z(i,T^*_{kj})-Z(i,T)|}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}$$

$$\geqslant\frac{\varepsilon}{2}\Big\}$$

$$\leqslant \sum_{|k|\leqslant \theta^{\varepsilon j}} \sum_{i=0}^{\theta^j} 2P\left\{\frac{|Z(i,T^*_{kj})-Z(i,T'_{kj})|}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}\geqslant\frac{\varepsilon}{2}\right\}$$

$$\leqslant 4\sum_{|k|\leqslant \theta^{\varepsilon j}} \sum_{i=0}^{\theta^j} \exp\left(-\frac{\varepsilon^2\log\theta^{j+1}}{4(\theta^2-1)}\right)$$

$$\leqslant 8\theta^{-2j}.$$

利用条件(3.5.39)和Slepian引理，我们得

$$(3.5.52)\ P\left\{\max_{0\leqslant i\leqslant\theta^j}\frac{Z(i,T^*_{kj})}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}\leqslant\frac{\theta}{1+\varepsilon}\right\}$$

$$\leqslant\prod_{i=0}^{[\theta^j]}P\left\{\frac{Z(i,T^*_{kj})}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}\leqslant\frac{\theta}{1+\varepsilon}\right\}$$

$$\leqslant\prod_{i=0}^{[\theta^j]}\left\{1-\exp\left(-\frac{\theta^2}{1+\varepsilon}\log\theta^{j+1}\right)\right\}$$

$$\leqslant\exp(-\theta^{\varepsilon j/4})\leqslant\theta^{-2j}.$$

所以，从(3.5.50)—(3.5.52)我们就得对每一充分大的$j$

$$(3.5.53)\ P\left\{\min_{|k|\leqslant\theta^{\varepsilon j}}\inf_{T\in A_{kj}}\max_{0\leqslant i\leqslant\theta^j}\frac{Y(i,T)}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}}\right.$$

$$\left.\leqslant\frac{1}{(1+\varepsilon)^2}\right\}$$

$$\leqslant 9\theta^{-2j}.$$

结合(3.5.47),(3.5.48),(3.5.53)及Borel-Cantelli引理得

$$(3.5.54)\ \varliminf_{T\to\infty}\sup_{|t|\leqslant1}\frac{|X(t+a_T,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}}\geqslant\frac{1}{(1+\varepsilon)^2}-\varepsilon$$

a.s.

由$\varepsilon$任意性得证(3.5.46)。定理3.5.1证毕。

**定理 3.5.2** (Csörgő, Lin and Shao, 1991)假设对$s_0=a_T$，$u_0=b_T$，(3.5.22),(3.5.23)和(3.5.24)被满足且对任何 $s>0$，$u>0$，$j\neq k$和$m\neq 1$

$$(3.5.55)\ EX(R(js,\ s,\ ku,\ u))X(R(ms,\ s,\ lu,\ u))\leqslant 0.$$

那么

$$(3.5.56)\quad \lim_{T\to\infty}\sup_{|t|\leq 1}\sup_{0\leq v\leq 1}\frac{|X(R(t,\ a_T,\ v,\ b_T))|}{H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2}}=1 \quad \text{a.s.}$$

$$(3.5.57)\quad \lim_{T\to\infty}\sup_{|t|\leq 1}\sup_{0\leq v\leq 1}\sup_{0\leq s\leq a_T}\sup_{0\leq u\leq b_T}|X(R(t,s,v,u))|/ (H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2})=1 \quad \text{a.s.}$$

**证** 首先，我们来证

$$(3.5.58)\quad \overline{\lim_{T\to\infty}}\sup_{|t|\leq 1}\sup_{0\leq v\leq 1}\sup_{0\leq s\leq a_T}\sup_{0\leq u\leq b_T}|X(R(t,s,v,u))|/ (H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2}) \leq 1 \quad \text{a.s.}$$

设$\theta>1$.定义$A_{kj}=\{T:\ \theta^{-(j+1)}<a_T\leq\theta^{-j},\ \theta^{-(k+1)}<b_T\leq\theta^{-k}\}$，$j$，$k=0$，1，….由条件(3.5.24)，当$\theta$充分接近1时有

$$(3.5.59)\quad \overline{\lim_{T\to\infty}}\sup_{|t|\leq 1}\sup_{0\leq v\leq 1}\sup_{0\leq s\leq a_T}\sup_{0\leq u\leq b_T}|X(R(t,s,v,u))|/ (H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2})$$
$$\leq\overline{\lim_{j\to\infty}}\sup_{k\geq 0}\sup_{T\in A_{kj}}\sup_{|t|\leq 1}\sup_{0\leq v\leq 1}\sup_{0\leq s\leq a_T}\sup_{0\leq u\leq b_T}|X(R(t, s,v,u))|/(H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2})$$
$$\leq\overline{\lim_{j\to\infty}}\sup_{k\geq 0}\sup_{|t|\leq 1}\sup_{0\leq v\leq 1}\sup_{0\leq s\leq \theta^{-j}}\sup_{0\leq u\leq\theta^{-k}}(1+\varepsilon)|X(R(t,s,v,u))|/(H_2(\theta^{-j},v,\theta^{-k})(2\log\theta^{j+k})^{1/2}).$$

利用引理3.5.3，我们推得

$$(3.5.60)\quad P\Big\{\sup_{k\geq 0}\sup_{|t|\leq 1}\sup_{0\leq v\leq 1}\sup_{0\leq s\leq\theta^{-j}}\sup_{0\leq u\leq\theta^{-k}}|X(R(t,s, v,u))|/(H_2(\theta^{-j},v,\theta^{-k})(2\log\theta^{j+k})^{1/2})\geq 1+\varepsilon\Big\}$$
$$\leq C\sum_{k=0}^{\infty}\theta^{j+k}\exp\{-(1+\varepsilon)^2\log\theta^{j+k}\}$$

$\leqslant C\theta^{-\varepsilon j}$.

从(3.5.59),(3.5.60),应用Borel-Cantelli引理得(3.5.58).

现在我们来证

$$(3.5.61)\ \varliminf_{T\to\infty}\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\frac{|X(R(t,a_T,v,b_T))|}{H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2}}\geqslant 1 \quad \text{a.s.}$$

注意到

$$(3.5.62)\ \varliminf_{T\to\infty}\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\frac{|X(R(t,a_T,v,b_T))|}{H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2}}$$

$$\geqslant\varliminf_{T\to\infty}\inf_{k\geqslant 0}\inf_{T\in A_{k,j}}\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\frac{|X(R(t,a_T,v,b_T))|}{H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2}}$$

$$\geqslant\varliminf_{T\to\infty}\inf_{k\geqslant 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant v\leqslant 1}\frac{|X(R(t,\theta^{-j},v,\theta^{-k}))|}{H_2(\theta^{-j},v,\theta^{-k})(2\log\theta^{j+k+2})^{1/2}}$$

$$-\varlimsup_{j\to\infty}\sup_{k\geqslant 0}\sup_{0\leqslant t\leqslant 1}\sup_{\theta^{-k-1}\leqslant v\leqslant \theta^{-k}+1}\sup_{0\leqslant s\leqslant\theta^{-j}}$$

$$\sup_{0\leqslant u\leqslant(1-\theta^{-1})\theta^{-k}}\frac{(1+\varepsilon)|X(R(t,s,v,u))|}{H_2(\theta^{-j},v-\theta^{-k-1},\theta^{-k})(2\log\theta^{j+k})^{1/2}}$$

$$-\varlimsup_{j\to\infty}\sup_{k\geqslant 0}\sup_{\theta^{-j-1}\leqslant t\leqslant\theta^{-j-1}+1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant u\leqslant\theta^{-k}}$$

$$\sup_{0\leqslant s\leqslant(1-\theta^{-1})\theta^{-j}}\frac{(1+\varepsilon)|X(R(t,s,v,u))|}{H_2(\theta^{-j},v,\theta^{-k})(2\log\theta^{j+k})^{1/2}}$$

$$\geqslant\varliminf_{j\to\infty}\inf_{k\geqslant 0}\max_{0\leqslant l\leqslant\theta^{j}}\max_{0\leqslant m\leqslant\theta^{k}}|X(R(l\theta^{-j},\theta^{-j},m\theta^{-k},\theta^{-k}))|/$$

$$(H_2(\theta^{-j},m\theta^{-k},\theta^{-k})(2\log\theta^{j+k+2})^{1/2})$$

$$-\varepsilon\varlimsup_{j\to\infty}\sup_{k\geqslant 0}\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant v\leqslant\theta^{-k-1}+1}\sup_{0\leqslant s\leqslant\theta^{-j}}$$

$$\sup_{0\leqslant u\leqslant(1-\theta^{-1})\theta^{-k}}\frac{(1+\varepsilon)|X(R(t,s,v,u))|}{H_2(\theta^{-j},v,(1-\theta^{-1})\theta^{-k})(2\log\theta^{j+k})^{1/2}}$$

$$
-\ \varepsilon \varlimsup_{j\to\infty} \sup_{k\geq 0} \sup_{0\leq t\leq \theta^{-j-1}+1} \sup_{0\leq v\leq 1} \sup_{0\leq s\leq (1-\theta^{-1})\theta^{-j}}
$$

$$
\sup_{0\leq u\leq \theta^{-k}} \frac{(1+\varepsilon)|X(R(t,s,v,u))|}{H_2((1-\theta^{-1})\theta^{-j},v,\theta^{-k})(2\log\theta^{j+k})^{1/2}}
$$

$$
=: I_1 - I_2 - I_3,
$$

这里条件(3.5.24)再一次被利用了.沿着(3.5.58)的证 明思路，回顾注3.5.1,我们得

$$
(3.5.63) \qquad I_2 + I_3 \leqslant 3\varepsilon \qquad \text{a.s.}
$$

对$I_1$，借助(3.5.55),我们可应用Slepian引理推得

$$
P\left\{\inf_{k\geq 0} \max_{0\leq l\leq \theta^{j}} \max_{0\leq m\leq \theta^{k}} \frac{|X(R(l\theta^{-j},\theta^{-j},m\theta^{-k},\theta^{-k}))|}{H_2(\theta^{-j},m\theta^{-k},\theta^{-k})(2\log\theta^{j+k+2})^{1/2}}\right.
$$

$$
\left.\leqslant 1-\varepsilon\right\}
$$

$$
\leqslant \sum_{k=0}^{\infty} \prod_{l=0}^{[\theta^j]} \prod_{m=0}^{[\theta^k]} P\left\{\frac{|X(R(l\theta^{-j},\theta^{-j},m\theta^{-k},\theta^{-k}))|}{H_2(\theta^{-j},m\theta^{-k},\theta^{-k})(2\log\theta^{j+k+2})^{1/2}} \leqslant 1-\varepsilon\right\}
$$

$$
\leqslant \sum_{k=0}^{\infty} \{1-\exp(-(1-\varepsilon)\log\theta^{j+k+2})\}^{\theta^{j+k}}
$$

$$
\leqslant \sum_{k=0}^{\infty} \exp(-\theta^{\varepsilon(j+k)}) \leqslant c\exp(-\theta^{\varepsilon j}).
$$

这就得证

$$
(3.5.64) \qquad I_1 \geqslant 1-\varepsilon \qquad \text{a.s.}
$$

结合(3.5.63),(3.5.64)和(3.5.62)得 (3.5.61). 定理3.5.2 证毕.

下述推论讨论了上面给出的例子.

**推论 3.5.1** 设$\{W(x, y);\ -\infty<x<\infty,\ 0\leqslant y<\infty\}$是标准两参数Wiener过程.那么

$$
\lim_{T\to\infty} \sup_{|t|\leq 1} \sup_{0\leq v\leq 1} \frac{|W(R(t,a_T,v,b_T))|}{(2a_Tb_T\log(1/a_Tb_T))^{1/2}} = 1 \qquad \text{a.s.}
$$

$$\lim_{T\to\infty}\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\sup_{0\leqslant u\leqslant b_T}\frac{|W(R(t,s,v,u))|}{(2a_Tb_T\log(1/a_Tb_T))^{1/2}}=1$$

a.s.

若附设

(3.5.65) $\log\log(Ta_T+(Ta_T)^{-1})=o(\log(1/a_T))\quad T\to\infty,$

那么

$$\lim_{T\to\infty}\sup_{|t|\leqslant 1}\frac{|W(t+a_T,T)-W(t,T)|}{(2Ta_T\log(1/a_T))^{1/2}}=1 \qquad \text{a.s.}$$

$$\lim_{T\to\infty}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\frac{|W(t+s,T)-W(t,T)|}{(2Ta_T\log(1/a_T))^{1/2}}=1 \qquad \text{a.s.}$$

(参见Csörgő和Révész (1981)定理1.14.2和定理S1.14.2)

**推论 3.5.2** 设$\{K(x,\ y);\ 0\leqslant x\leqslant 1,\ 0\leqslant y<\infty\}$是Kiefer过程.

那么

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant 1-a_T}\sup_{0\leqslant v\leqslant 1}\frac{|K(R(t,a_T,v,b_T))|}{(2a_T(1-a_T)b_T\log(1/a_Tb_T))^{1/2}}=1 \text{ a.s.}$$

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant 1-a_T}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\sup_{0\leqslant u\leqslant b_T}|K(R(t,s,v,u))|/((2a_T(1$$

$$-a_T)b_T\log(1/a_Tb_T))^{1/2})=1 \qquad \text{a.s.}$$

若附设(3.5.65)被满足，那么

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant 1-a_T}\frac{|K(t+a_T,T)-K(t,T)|}{(2Ta_T(1-a_T)\log(1/a_T))^{1/2}}=1 \qquad \text{a.s.}$$

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant 1-a_T}\sup_{0\leqslant s\leqslant a_T}\frac{|K(t+s,\ T)-K(t,\ T)|}{(2Ta_T(1-a_T)\log(1/a_T))^{1/2}}=1 \text{ a.s.}$$

**推论 3.5.3** 设$\{X(t,\ v);\ -\infty<t<\infty,\ 0\leqslant v<\infty\}$是如例3中的两参数Ornstein-Uhlenbeck过程.假设存在$C_0>0$使得对任一$0<s\leqslant a_T$

$$\int_{0<x\leqslant T,\ \lambda(x)\geqslant 1/s}\frac{\gamma(x)}{\lambda(x)}dx\leqslant C_0s\int_{0<x\leqslant T,\ \lambda(x)\leqslant 1/s}\gamma(x)dx$$

且

$$\log\log(H_1(a_T,\ T) + H_1^{-1}(a_T,\ T)) = o(\log(1/a_T))$$

当 $T\to\infty$，

其中

$$H_1^2(a_T,T) = 2\int_0^T \frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)a_T))dx.$$

那么

$$\lim_{T\to\infty} \sup_{0\leqslant t\leqslant 1} \frac{|X(t+a_T,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}} = 1 \quad \text{a.s.}$$

$$\lim_{T\to\infty} \sup_{0\leqslant t\leqslant 1} \sup_{0\leqslant s\leqslant a_T} \frac{|X(t+s,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}} = 1 \quad \text{a.s.}$$

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# 主 题 索 引

# Strong Limit Theorems

Mathematics and Its Applications (Chinese Series)

# Strong Limit Theorems

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## SERIES EDITOR'S PREFACE

'Et moi, ..., si j'avait su comment en revenir, je n'y serais point allé.'

Jules Verne

The series is divergent; therefore we may be able to do something with it.

O. Heaviside

One service mathematics has rendered the human race. It has put common sense back where it belongs, on the topmost shelf next to the dusty canister labelled 'discarded nonsense'.

Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and nonlinearities abound. Similarly, all kinds of parts of mathematics serve as tools for other parts and for other sciences.

Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...'; 'One service logic has rendered computer science ...'; 'One service category theory has rendered mathematics ...'. All arguably true. And all statements obtainable this way form part of the raison d'être of this series.

This series, *Mathematics and Its Applications*, started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote

> "Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related. Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as 'experimental mathematics', 'CFD', 'completely integrable systems', 'chaos, synergetics and large-scale order', which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."

By and large, all this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find, see, and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such books available.

If anything, the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces, algebraic geometry, modular functions, knots, quantum field theory, Kac-Moody algebras, monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry'; a combination of words which sounds like it might not even exist, let alone be applicable. And yet it is being applied: to statistics via designs, to radar/sonar detection arrays (via finite projective planes), and to bus connections of VLSI chips (via difference sets). There seems to be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And, accordingly, the applied mathematician needs to be aware of much more. Besides analysis and numerics, the traditional workhorses, he may need all kinds of combinatorics, algebra, probability, and so on.

In addition, the applied scientist needs to cope increasingly with the nonlinear world and the extra

mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the nonlinear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis, superspace and anticommuting integration, $p$-adic and ultrametric space. All three have applications in both electrical engineering and physics. Once, complex numbers were equally outlandish, but they frequently proved the shortest path between 'real' results. Similarly, the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading - fortunately.

Thus the original scope of the series, which for various (sound) reasons now comprises five subseries: white (Japan), yellow (China), red (USSR), blue (Eastern Europe), and green (everything else), still applies. It has been enlarged a bit to include books treating of the tools from one subdiscipline which are used in others. Thus the series still aims at books dealing with:

- a central concept which plays an important role in several different mathematical and/or scientific specialization areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have, and have had, on the development of another.

To quite a large extent limit theorems (or laws of large numbers) are what makes statistics applicable. Particularly important are strong limit theorems. This is well known. Still I was surprised to find that 2651 articles and books have appeared with the phrase 'strong limit theorems' in title, abstract, keyword list, or table of contents. And most of these appeared after 1981, which appears to be the last time that the field was surveyed as a whole.

In this volume the authors survey the results obtained in strong limit theory in the last 10 years or so, including some of their own work. It seems to me a most useful thing to have around for every mathematician or statistician who just might need a strong limit related result that goes beyond the standard ones.

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard

La physique ne nous donne pas seulement l'occasion de résoudre des problèmes ... elle nous fait pressentir la solution.

H. Poincaré

Never lend books, for no one ever returns them; the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people, but to be wrong for more sophisticated reasons.

David Butler

Bussum, 10 February 1992

Michiel Hazewinkel

# Preface

Strong approximation and strong convergence methodologies have been very active areas of research in probability theory during the past two decades. The 1981 monograph of M.Csörgő and P.Révész *Strong Approximations in Probability and Statistics* sums up the basic results of this kind up to the beginning of the 1980s. In the introduction to their book, the authors pointed out possible directions for further research, such as multitime parameter processes and processes in higher-dimensional Euclidean space (or Banach space), as well as the case of nonindependent and /or non-identically distributed random variables.

In recent years, many results of Csörgő and Révész ( 1981 ) have been generalized and improved to a great extent. Limit properties of a Wiener process have been continuously studied extensively and deeply by some authors, and limit results of the increments of partial sums of a sequence of random variables have been sharpened and generalized to the case of independent, but not necessarily identically distributed random variables by Z.Y. Lin and Q.M. Shao. Path properties of some Gaussian processes related to a Wiener process have been also investigated. This monograph aims to provide an overview of this work.

We are deeply indebted to Professor M. Csörgő (Carleton University, Canada) and can only say that this monograph could never have been written without his encouragement. Our best thanks are due to Dr. Q. M. Shao whose comments greatly improved the book. Our thanks go also to B. Chen, Z. W. Cai and Z. G. Su for their helpful suggestions.

Lin Zhengyan

Lu Chuanrong

Hangzhou University, April 1992

# Contents

Chapter 1

# The Increments of a Wiener and Related Gaussian Processes

The results of the increments of a Wiener process and some related Gaussian processes deeply describe the properties of sample paths. They are the important achievements in probability theory in the last few decades. Many authors have studied these subjects following the 1981 monograph of M. Csörgő and P. Révész : *Strong Approximations in Probability and Statistics.* In this chapter, we will introduce some new advances in this area.

## 1.1 How Large Are the Increments of a Wiener Process?

### 1.1.1 Csörgő-Révész's increments

Let $\{ W(t) ;\ 0 \leqslant t < \infty \}$ be a standard Wiener process on a probability space $(\Omega, \mathscr{F}, P)$. Csörgő and Révész proved the following theorem first.

**Theorem 1.1.1** (Csörgő, Révész 1979) *Let* $0 < a_T \leqslant T$ *be a function of* $T$ *for which*

(i) $a_T$ *is monotonically non-decreasing,*

(ii) $T/a_T$ *is montonically non-decreasing.*

*Then*

$$\varlimsup_{T \to \infty} \sup_{0 \leqslant t \leqslant T-a_T} \sup_{0 \leqslant s \leqslant a_T} \beta_T | W(t+s) - W(t) | = 1 \qquad \text{a.s.} \quad (1.1.1)$$

$$\varlimsup_{T \to \infty} \sup_{0 \leqslant t \leqslant T-a_T} \beta_T | W(t+a_T) - W(t) | = 1 \qquad \text{a.s.} \quad (1.1.2)$$

$$\overline{\lim_{T \to \infty}} \sup_{0 \leqslant s \leqslant a_T} \beta_T | W(T+s) - W(T)| = 1 \quad \text{a.s.} \tag{1.1.3}$$

$$\overline{\lim_{T \to \infty}} \beta_T | W(T+a_T) - W(T)| = 1 \quad \text{a.s.} \tag{1.1.4}$$

*where*

$$\beta_T = \{ 2 a_T (\log (T/a_T) + \log \log T) \}^{-1/2 (*)}.$$

*If we have also*

(iii) $\lim_{T \to \infty} (\log T/a_T)/\log \log T = \infty$ ,

*then*

$$\lim_{T \to \infty} \sup_{0 \leqslant t \leqslant T-a_T} \sup_{0 \leqslant s \leqslant a_T} \beta_T | W(t+s) - W(t)| = 1 \quad \text{a.s.} \tag{1.1.5}$$

$$\lim_{T \to \infty} \sup_{0 \leqslant t \leqslant T-a_T} \beta_T | W(t+a_T) - W(t)| = 1 \quad \text{a.s.} \tag{1.1.6}$$

Deo (1977) showed that if condition (iii) fails, then

$$\underline{\lim_{T \to \infty}} \sup_{0 \leqslant t \leqslant T-a_T} \sup_{0 \leqslant s \leqslant a_T} \beta_T | W(t+s) - W(t)| < 1 \quad \text{a.s.}$$

provided $\overline{\lim_{T \to \infty}} (\log T/a_T)/\log \log T < \infty$ . This suggests the following problem : find the normalizing factor $\delta_T = \delta_T(a_T)$ such that

$$\underline{\lim_{T \to \infty}} \sup_{0 \leqslant t \leqslant T-a_T} \sup_{0 \leqslant s \leqslant a_T} \delta_T | W(t+s) - W(t)| = 1 \quad \text{a.s.} \tag{1.1.7}$$

$$\underline{\lim_{T \to \infty}} \sup_{0 \leqslant t \leqslant T-a_T} \delta_T | W(t+a_T) - W(t)| = 1 \quad \text{a.s.} \tag{1.1.8}$$

Some partial answers concerning (1.1.7) and (1.1.8) were given by Book and Shore (1978), Csáki and Révész (1979) and Shao (1986) et al. Two of these are as follows :

1. (Book, Shore 1978) Let $a_T$ be as in Theorem 1.1.1. If

(iv) $\lim_{T \to \infty} (\log T/a_T)/\log \log T = r \quad 0 \leqslant r \leqslant \infty$ ,

then

$$\underline{\lim_{T \to \infty}} \sup_{0 \leqslant t \leqslant T-a_T} \beta_T | W(t+a_T) - W(t)| = \left( \frac{r}{1+r} \right)^{1/2} \quad \text{a.s.} \tag{1.1.9}$$

2. (Csáki, Révész 1979) Let $a_T$ be as in Theorem 1.1.1. Then

---

(*) Here, and in the sequel, we shall define $\log t = \log(\max(t, 1))$, $\log \log t = \log \log (\max(t, e))$.

$$18^{-1} \leqslant \varlimsup_{T\to\infty} \gamma_1(T) \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} |W(t+s)-W(t)| \leqslant 46 \quad \text{a.s.}$$

where

$$\gamma_1(T)=\{ 2\, a_T \log(1+\frac{\pi^2}{16}\ \frac{T}{a_T \log\log T})\}^{-1/2}.$$

Furthermore, if

(v) $\lim_{T\to\infty} (\log T/a_T)/\log\log\log T=\infty$,

then

$$\varlimsup_{T\to\infty} \gamma_1(T) \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} |W(t+s)-W(t)|=1 \quad \text{a.s.} \tag{1.1.10}$$

Shao (1986)weakened the condition (v)and obtained the following result.

**Theorem 1.1.2** (Shao 1986) *Let $a_T$ be as in Theorem* 1.1.1. *If*

(vi) $\lim_{T\to\infty} (T/a_T)/\log\log T=\infty$ ,

*then*

$$\varliminf_{T\to\infty} \gamma(T) \sup_{0\leqslant t\leqslant T-a_T} |W(t+a_T)-W(t)|=1 \quad \text{a.s.} \tag{1.1.11}$$

$$\varliminf_{T\to\infty} \gamma(T) \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} |W(t+s)-W(t)|=1 \quad \text{a.s.} \tag{1.1.12}$$

*where*

$$\gamma(T)=\{ 2a_T(\log T/a_T-\log\log\log T)\}^{-1/2}.$$

In order to prove this theorem, we need the following lemmas.

**Lemma 1.1.1** (Slepian 1962, Adler 1989) *Let $\{X(t);\ t\in T\}$ and $\{Y(t);\ t\in T\}$ be centered Gaussian processes such that $EX^2(t)=EY^2(t)$ for all $t\in T$ and $EY(t)Y(s)\leqslant EX(t)X(s)$ for all $s, t\in T$. Then*

$$P\{\sup_{t\in T} X(t)\leqslant u\}\geqslant P\{\sup_{t\in T} Y(t)\leqslant u\}.$$

**Lemma 1.1.2** (Révész 1982) *Let $k$ be an arbitrary positive number. Then for any $\varepsilon>0$ there exists a $u_0=u_0(k,\varepsilon)>0$ such that, for any $u\geqslant u_0$, we have*

$$(1-\varepsilon)\frac{ku}{\sqrt{2\pi}}e^{-u^2/2}\leqslant P\{\sup_{0\leqslant x\leqslant h}(W(x+1)-W(x))>u\} \tag{1.1.13}$$
$$\leqslant P\{\sup_{0\leqslant x\leqslant k}\sup_{0\leqslant s\leqslant 1}(W(x+s)-W(x))>u\}\leqslant c\frac{ku}{\sqrt{2\pi}}e^{-u^2/2},$$

*where the constant* $c<25$.

*Proof* The first inequality in (1.1.13) is well-known (see Qualls and Watanabe 1972). We need only to prove the last inequality. Let

$$x_i=i/u^2,\qquad i=1,2,\cdots,[u^2k]^{(*)}$$

be a partition of the interval $[0,k]$ and define the events

$$B_i=\{\sup_{0\leqslant s\leqslant u^{-2}}(W(x_i-s)-W(x_i))>1\},$$
$$A_i(v)=\{\sup_{0\leqslant s\leqslant 1}(W(x_i+s)-W(x_i))\geqslant u-v/u,$$
$$(v-\Delta v)/u\leqslant\sup_{0\leqslant s\leqslant u^{-2}}(W(x_i-s)-W(x_i))\leqslant v/u\}.$$

Then as $u\to\infty$ and $\Delta v\to 0$, we have

$$P\{A_i(v)\}\approx\frac{2}{\sqrt{2\pi}}\exp(-\frac{v^2}{2})\cdot\frac{2}{\sqrt{2\pi}}\frac{1}{u-v/u}\exp(-\frac{1}{2}(u-\frac{v}{u})^2\Delta v,$$
$$P\{B_i\}\approx\frac{2}{u\sqrt{2\pi}}\exp(-\frac{u^2}{2}).$$

Therefore there exists an absolute constant $u_0>0$ such that for $u>u_0$, we have

$$P\{\sup_{0\leqslant x\leqslant k}\sup_{0\leqslant s\leqslant 1}(W(x+s)-W(x))>u\}$$
$$\leqslant[u^2k]\left\{\frac{2}{\pi}\int_0^u\frac{1}{u-v/u}\exp(-\frac{v^2}{2})\exp(-\frac{1}{2}(u-\frac{v}{u})^2)dv\right.$$
$$\left.+\frac{2}{u\sqrt{2\pi}}\exp(-\frac{u^2}{2})\right\}\leqslant c\frac{ku}{\sqrt{2\pi}}\exp(-\frac{u^2}{2}),$$

where $c<25$. The proof is completed.

**Lemma 1.1.3** (Révész 1982) *For any* $\varepsilon>0$ *there exists* $u=u_0(\varepsilon)>0$ *and* $T_0=T_0(\varepsilon)>0$ *such that*

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(*)In this book, the sign $[\cdot]$ sometimes denotes the greatest integer part or at other times denotes brackets. It will be clear from the context.

$$\exp\{-25\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2}\}\leqslant P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant 1}(W(t+s)-W(t))\leqslant u\}\quad(1.1.14)$$

$$\leqslant P\{\sup_{0\leqslant t\leqslant T}(W(t+1)-W(t))\leqslant u\}\leqslant\exp\{-(1-\varepsilon)\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2}\}$$

*for* $u\geqslant u_0$ *and* $T\geqslant T_0$.

*Proof* Let $k=[T]$. From Lemma 1.1.1 and Lemma 1.1.2, we have

$$P\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant 1}(W(t+s)-W(t))\leqslant u\}$$
$$\geqslant P\{\max_{0\leqslant i\leqslant k}\sup_{i\leqslant t\leqslant i+1}\sup_{0\leqslant s\leqslant 1}(W(t+s)-W(t))\leqslant u\}$$
$$\geqslant(1-P\{\sup_{0\leqslant t\leqslant 1}\sup_{0\leqslant s<1}(W(t+s)-W(t))>u\})^{k+1}$$
$$\geqslant(1-\frac{cu}{\sqrt{2\pi}}e^{-u^2/2})^{k+1}\geqslant\exp(-25\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2})$$

for $u>u_0$, which proves the first inequality of (1.1.14). We now prove the last inequality of (1.1.14). If $k<T$ is a positive integer, then

$$\sup_{0\leqslant t\leqslant T}(W(t+1)-W(t)),\geqslant\max_{0\leqslant i\leqslant l}\sup_{i(k+1)\leqslant t<(i+1)(k+1)}(W(t+1)-W(t)),$$

where $l$ is the largest integer for which $(l+1)(k+1)-1\leqslant T$. It is easy to see that $\{\sup_{i(k+1)\leqslant t<(i+1)(k+1)}(W(t+1)-W(t));i=0,1,\cdots,l\}$ are independent. Hence, by Lemma 1.1.2, we have

$$P\{\sup_{0\leqslant t\leqslant T}(W(t+1)-W(t))\leqslant u\}$$
$$\leqslant P\{\max_{0\leqslant i\leqslant l}\sup_{i(k+1)\leqslant t<(i+1)(k+1)}(W(t+1)-W(t))\leqslant u\}$$
$$\leqslant(P\{\sup_{0\leqslant t\leqslant k}(W(t+1)-W(t))\leqslant u\})^{l+1}$$
$$\leqslant(1-\frac{1+o(1)}{\sqrt{2\pi}}ku\,e^{-u^2/2})^{l+1}$$
$$\leqslant\exp\{-(1+o(1))\frac{k(l+1)u}{\sqrt{2\pi}}e^{-u^2/2}\}$$
$$\leqslant\exp\{-(1-\varepsilon)\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2}\}$$

for $T\geqslant T_0(\varepsilon)$ and $u\geqslant u_0(\varepsilon)$, which proves Lemma 1.1.3.

From the proof of Lemma 1.1.3, we have :

**Lemma 1.1.4** (Révész 1982) *For any $\varepsilon>0$ there exists $u_0=u_0(\varepsilon)$ and $T_0=T_0(\varepsilon)$ such that*

$$\exp\left\{-50\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2}\right\}\leqslant P\left\{\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant 1}|W(t+s)-W(t)|\leqslant u\right\}$$

$$\leqslant P\left\{\sup_{0\leqslant t\leqslant T}|W(t+1)-W(t)|\leqslant u\right\}\leqslant\exp\left\{-2(1-\varepsilon)\frac{Tu}{\sqrt{2\pi}}e^{-u^2/2}\right\}$$

*for $T\geqslant T_0$ and $u\geqslant u_0$.*

We can write the following obvious fact.

**Lemma 1.1.5** *Let $\{\xi,\xi_n;n\geqslant 1\}$ be a sequence of random variables. If*

$$P\{\xi_n\geqslant\xi\}\to 0 \quad \text{as} \quad n\to\infty,$$

*then there is a subsequence $\{\xi_{n_k}\}$ such that*

$$\overline{\lim_{k\to\infty}}\ \xi_{n_k}\leqslant\xi \qquad \text{a.s.}$$

*So that*

$$\underline{\lim_{n\to\infty}}\ \xi_n\leqslant\xi \qquad \text{a.s.}$$

*Proof of Theorem* 1.1.2 .

1° We prove that

$$\underline{\lim_{T\to\infty}}\ \gamma(T)\sup_{0\leqslant t\leqslant T-a_T}|W(t+a_T)-W(t)|\geqslant 1 \quad \text{a.s.} \qquad (1.1.15)$$

Taking $\varepsilon=1/2$ in Lemma 1.1.4 and noting condition (vi ), we have

$$P\left\{\sup_{0\leqslant t\leqslant T-a_T}|W(t+a_T)-W(t)|<\gamma^{-1}(T)\right\}$$

$$\leqslant\exp\left\{-\frac{T}{a_T\sqrt{2\pi}}\sqrt{2\log\frac{T}{a_T\log\log T}}\cdot\frac{a_T\log\log T}{T}\right\}$$

$$\leqslant(\log T)^{-4}$$

for large $T$. Denote $T_k=k^{\sqrt{k}}$ $(k=1,2,\cdots)$. By the Borel-Cantelli lemma,

we obtain

$$\varlimsup_{k\to\infty} \gamma(T_k) \sup_{0\leqslant t\leqslant T_k-a_{T_k}} |W(t+a_{T_k})-W(t)| \geqslant 1 \qquad \text{a.s.} \quad (1.1.16)$$

If $T_k \leqslant T \leqslant T_{k+1}$, then we have

$$\gamma(T) \sup_{0\leqslant t\leqslant T-a_T} |W(t+a_T)-W(t)|$$

$$\geqslant (2a_{T_{k+1}} \log \frac{T_{k+1}/a_{T_{k+1}}}{\log\log T_k})^{-1/2} ( \sup_{0\leqslant t\leqslant T_k-a_{T_k}} |W(t+a_{T_k})-W(t)|$$

$$- \sup_{0\leqslant t\leqslant T_k-a_{T_k}} \sup_{0\leqslant s\leqslant a_{T_{k+1}}-a_{T_k}} |W(t+s)-W(t)|)$$

$$=: A_k\gamma(T_k)I(T_k)-Z_k(T_k), \qquad (1.1.17)$$

where

$$A_k=\left(\frac{T_k}{T_{k+1}} \cdot \frac{(a_{T_k}/T_k)\log((T_k/a_{T_k})/\log\log T_k)}{(a_{T_{k+1}}/T_{k+1})\log((T_{k+1}/a_{T_{k+1}})/\log\log T_k)}\right)^{1/2}.$$

Note that $x\log(1/xa)$ is a monotonically increasing function of $x$ when $eax \leqslant 1$ and $a>0$. Therefore

$$1\geqslant \lim_{k\to\infty} A_k \geqslant \lim_{k\to\infty} (T_k/T_{k+1})^{1/2}=1. \qquad (1.1.18)$$

On the other hand, by Theorem 1.1.1, we have

$$\varlimsup_{k\to\infty} \sup_{0\leqslant t\leqslant T_k-a_{T_k}} \sup_{0\leqslant s\leqslant a_{T_{k+1}}-a_{T_k}} \beta_1(T_k)|W(t+s)-W(t)|\leqslant 1 \qquad \text{a.s.}$$

where

$$\beta_1(T_k)=\{2(a_{T_{k+1}}-a_{T_k})(\log \frac{T_k+a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}+\log\log(T_k+a_{T_{k+1}}))\}^{-1/2}.$$

It is easy to see that for large $k$, we have also

$$a_{T_{k+1}}-a_{T_k}\leqslant a_{T_{k+1}}(1-T_k/T_{k+1})\leqslant 6\,a_{T_{k+1}}/k^{1/3},$$

which implies

$$\beta_1^{-2}(T_k)(2a_{T_{k+1}}\log((T_{k+1}/a_{T_{k+1}})/\log\log T_k))^{-1}$$

$$\leqslant \frac{6}{k^{1/3}}(\log(\frac{T_k+a_{T_{k+1}}}{6a_{T_{k+1}}}k^{1/3})+\log\log(T_k+a_{T_{k+1}}))/\log\left(\frac{T_{k+1}/a_{T_{k+1}}}{\log\log T_k}\right)$$

$$\to 0 \text{ as } k\to\infty.$$

Therefore

$$\overline{\lim_{k\to\infty}} Z_k(T_k)=0 \qquad \text{a.s.}$$

Combining this with (1.1.16)—(1.1.18) yields (1.1.15).

2° We prove that

$$\underline{\lim}_{T\to\infty} \gamma(T) \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |W(t+s)-W(t)|\le 1 \qquad \text{a.s.} \qquad (1.1.19)$$

If $\overline{\lim\limits_{T\to\infty}}$ $(\log T/a_T)/\log\log\log T=\infty$, there exists a sequence of positive numbers $\{T_N\}$ such that

$$\lim_{N\to\infty} (\log T_N/a_{T_N})/\log\log\log T_N=\infty. \qquad (1.1.20)$$

For any given $\varepsilon>0$, by using Lemma 1.2.1 in Csörgő and Révész (1981) we have

$$P\{(2a_{T_N}\log T_N/a_{T_N})^{-1/2} \sup_{0\le t\le T_N-a_{T_N}} \sup_{0\le s\le a_{T_N}} |W(t+s)-W(t)|\ge 1+\varepsilon\}$$

$$\le C\frac{T_N}{a_{T_N}}\exp(-(1+\varepsilon)\log\frac{T_N}{a_{T_N}})=C(\frac{a_{T_N}}{T_N})^{\varepsilon}\to 0 \qquad (N\to\infty).$$

Hence from Lemma 1.1.5 we have

$$\underline{\lim}_{N\to\infty} (2a_{T_N}\log T_N/a_{T_N})^{-1/2} \sup_{0\le t\le T_N-a_{T_N}} \sup_{0\le s\le a_{T_N}} |W(t+s)-W(t)|\le 1 \qquad \text{a.s.}$$

From (1.1.20)

$$\lim_{N\to\infty} (2a_{T_N}\log T_N/a_{T_N})^{-1/2}/\gamma(T_N)=1.$$

It follows that (1.1.19) holds true.

If $\overline{\lim\limits_{T\to\infty}}$ $(\log T/a_T)/\log\log\log T<\infty$, i.e. there exists a constant $c_0>0$ such that

$$T/a_T\le(\log\log T)^{c_0}. \qquad (1.1.21)$$

Let $T_k=e^{k^2}$ $(k=2,3,\cdots)$. By using Lemma 1.1.4 and by condition (vi), for any $\varepsilon>0$, we have

$$P\{\sup_{T_k\le t\le T_{k+1}-a_{T_{k+1}}} \sup_{0\le s\le a_{T_{k+1}}} |W(t+s)-W(t)|\le(1+\varepsilon)\gamma^{-1}(T_{k+1})\}$$

$$\geqslant \exp\left\{ -\frac{100}{\sqrt{2\pi}} \frac{T_{k+1}-T_k}{a_{T_{k+1}}} \frac{(1+\varepsilon)\sqrt{2\log(T_{k+1}/a_{T_{k+1}}\log\log T_{k+1})}}{(T_{k+1}/a_{T_{k+1}})^{1+\varepsilon}} \right.$$
$$\left. \cdot (\log\log T_{k+1})^{1+\varepsilon} \right\} \geqslant k^{-2/3}$$

for large $k$. By the Borel-Cantelli lemma, it follows that

$$\varlimsup_{k\to\infty} \gamma(T_{k+1}) \sup_{T_k \leqslant t \leqslant T_{k+1}-a_{T_{k+1}}} \sup_{0\leqslant s\leqslant a_{T_{k+1}}} |W(t+s)-W(t)| \leqslant 1+\varepsilon \quad \text{a.s.} \tag{1.1.22}$$

Note that

$$\sup_{0\leqslant t\leqslant T_{k+1}-a_{T_{k+1}}} \sup_{0\leqslant s\leqslant a_{T_{k+1}}} |W(t+s)-W(t)| \tag{1.1.23}$$

$$\leqslant \sup_{T_k\leqslant t\leqslant T_{k+1}-a_{T_{k+1}}} \sup_{0\leqslant s\leqslant a_{T_{k+1}}} |W(t+s)-W(t)| + \sup_{0\leqslant u<v\leqslant T_k} |W(v)-W(u)|.$$

By the well-known law of the iterated logarithm

$$\varlimsup_{k\to\infty} \sup_{0\leqslant u\leqslant v\leqslant T_k} (2T_k \log\log T_k)^{-1/2} |W(v)-W(u)| \leqslant 1 \quad \text{a.s.}$$

From (1.1.21) and condition (vi), we have

$$\gamma(T_{k+1})(2T_k\log\log T_k)^{1/2} \to 0 \quad \text{as} \quad k\to\infty.$$

Therefore

$$\varlimsup_{k\to\infty} \gamma(T_{k+1}) \sup_{0\leqslant u<v\leqslant T_k} |W(v)-W(u)| = 0 \quad \text{a.s.} \tag{1.1.24}$$

So, by merging (1.1.22)—(1.1.24), (1.1.19) holds true. The proof is completed.

### 1.1.2 Lag increments

Another form of increments, lag increments of a Wiener process were presented and discussed by Hanson and Russo (1983a). Chen, Kong and Lin (1986) sharpened their results and proved

**Theorem 1.1.3** (Chen, Kong, Lin 1986)

$$\varlimsup_{T\to\infty} \sup_{0<t\leqslant T} |W(T)-W(T-t)|/d(T,t) = 1 \quad \text{a.s.} \tag{1.1.25}$$

$$\lim_{T\to\infty} \sup_{0<t\leqslant T} \sup_{t\leqslant s\leqslant T} |W(s)-W(s-t)|/d(T,t)=1 \quad \text{a.s.} \tag{1.1.26}$$

$$\lim_{T\to\infty} \sup_{0<t\leqslant T} \sup_{0\leqslant s\leqslant t} |W(T)-W(T-s)|/d(T,t)=1 \quad \text{a.s.} \tag{1.1.27}$$

*where*

$$d(T,t)=\{2t(\log T/t+\log\log t)\}^{1/2}.$$

*Proof* 1° Using the law of the iterated logarithm, we have the left hand side of (1.1.25)

$$\geqslant \overline{\lim_{T\to\infty}} |W(T)|/(2T\log\log T)^{1/2}=1 \quad \text{a.s.} \tag{1.1.28}$$

2° In order to prove that

$$\overline{\lim_{T\to\infty}} \sup_{0<t\leqslant T} \sup_{t\leqslant s\leqslant T} |W(s)-W(s-t)|/d(T,t)\leqslant 1 \quad \text{a.s.} \tag{1.1.29}$$

we take the real number $\theta>1$ so that $1<2(1+\varepsilon)^2/((2+\varepsilon)\theta)=:1+2\varepsilon'$ for some $\varepsilon>0$. For $n=1,2,\cdots$ and $k=\cdots,-2,-1,0,1,\cdots,k_n$, denote $T_n=2^n$, $t_k=\theta^k$, where $k_n=[(n+1)\log 2/\log\theta]+1$. Write $k_0=[1/\log\theta]$, $k'_n=[\log(T_{n+1}/(\log T_n)^{1/\varepsilon'})/\log\theta]$.

When $T_n\leqslant T\leqslant T_{n+1}$, we have

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T} \sup_{t\leqslant s\leqslant T} |W(s)-W(s-t)|/d(T,t)\\
&\leqslant \sup_{-\infty\leqslant k\leqslant k_n-1} \sup_{t_k\leqslant t\leqslant t_{k+1}} \sup_{t\leqslant s\leqslant T_{n+1}} \frac{|W(s)-W(s-t)|}{\{2t_k(\log T_n/t_{k+1}+\log\log t_k)\}^{1/2}}\\
&=: \sup_{-\infty<k\leqslant k_n-1} A_{nk}.
\end{aligned} \tag{1.1.30}$$

An inspection of the proof of the Csörgő-Révész lemmas (see Lemmas 1.1.1 and 1.2.1 in Csörgő and Révész 1981) convinces one that for any $0<T$, $0<v$, $0<h\leqslant T$, we have

$$P\{\sup_{0\leqslant s'\leqslant s\leqslant T,\, s-s'\leqslant h} h^{-1/2}|W(s)-W(s')|\geqslant v\}\leqslant \frac{CT}{h}\exp\left(-\frac{v^2}{2+\varepsilon}\right), \tag{1.1.31}$$

where $C$ is a positive constant depending only on $\varepsilon$. Using this inequality, for $-\infty<k\leqslant k_0$, we have

$$\begin{aligned}
&P\{A_{nk}\geqslant 1+\varepsilon\}\\
&\leqslant P\{\sup_{0\leqslant s-t,\, s\leqslant T_{n+1},\, 0<t\leqslant t_{k+1}} t_{k+1}^{-1/2}|W(s)-W(s-t)|
\end{aligned}$$

$$\geqslant (1+\varepsilon)\left(\frac{2t_k}{t_{k+1}}\log\frac{T_n}{t_{k+1}}\right)^{1/2}\Bigg\}$$

$$\leqslant c\frac{T_{n+1}}{t_{k+1}}\exp\left\{-\frac{2(1+\varepsilon)^2}{(2+\varepsilon)\theta}\log\frac{T_n}{t_{k+1}}\right\}$$

$$= c\frac{T_{n+1}}{t_{k+1}}\left(\frac{t_{k+1}}{T_n}\right)^{1+2\varepsilon'} = c(\theta^{k+1}/2^n)2\varepsilon'. \qquad (1.1.32)$$

(Here, and in the sequel, $c$ denotes a positive constant, which may take different values at different places.) Hence, it follows that

$$\sum_{n=1}^{\infty}\sum_{-\infty<k\leqslant k_\theta} P\{A_{nk}\geqslant 1+\varepsilon\}$$

$$\leqslant c\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(\theta^{-k}2^{-n})^{2\varepsilon'} + c\sum_{n=1}^{\infty}(k_\theta+1)(\theta e)^{2\varepsilon'}2^{-2\varepsilon' n} < \infty. \qquad (1.1.33)$$

For the case $k_\theta<k\leqslant k_n-1$, using the inequality (1.1.31) again, we have

$$P\{A_{nk}\geqslant 1+\varepsilon\}$$

$$\leqslant P\Bigg\{\sup_{0\leqslant s-t,\, s\leqslant T_{n+1}}\ \sup_{0\leqslant t\leqslant t_{k+1}} t_{k+1}^{-1/2}|W(s)-W(s-t)|$$

$$\geqslant (1+\varepsilon)\left(\frac{2t_k}{t_{k+1}}\left(\log\frac{T_n}{t_{k+1}}+\log\log t_k\right)\right)^{1/2}\Bigg\}$$

$$\leqslant c(T_{n+1}/t_{k+1})\exp\{-(2(1+\varepsilon)^2/(2+\varepsilon)\theta)\log((T_n\log t_k)/t_{k+1})\}$$

$$= c(t_{k+1}/T_{n+1})^{2\varepsilon'}(\log t_k)^{-(1+2\varepsilon')}. \qquad (1.1.34)$$

Note that when $k_\theta<k\leqslant k_n'$, we have

$$(t_{k+1})^{2\varepsilon'} \leqslant \theta^{(k_n'+1)2\varepsilon'} \leqslant \left(\frac{\theta T_{n+1}}{(\log T_n)^{1/\varepsilon'}}\right)^{2\varepsilon'} = \frac{\theta^{2\varepsilon'}T_{n+1}^{2\varepsilon'}}{(\log T_n)^2}.$$

From (1.1.34), it follows that

$$\sum_{n=1}^{\infty}\sum_{k=k_\theta+1}^{k_n'} P\{A_{nk}\geqslant 1+\varepsilon\}$$

$$\leqslant c\sum_{n=1}^{\infty}\frac{\theta^{2\varepsilon'}}{(\log T_n)^2}\sum_{k=k_\theta+1}^{k_n'}(\log t_k)^{-(1+2\varepsilon')}$$

$$\leqslant c\sum_{n=1}^{\infty} n^{-2} \sum_{k=1}^{\infty} k^{-(1+2\varepsilon')} < \infty . \tag{1.1.35}$$

For the case $k_n' < k \leqslant k_n - 1$, we have

$$T_n^{1/2} \leqslant t_{k+1} \leqslant \theta T_{n+1},$$

$$k_n - k_n' \leqslant (\varepsilon' \log\theta)^{-1} \log\log 2^n + 2 =: k_n''.$$

Using (1.1.34) again, one sees that

$$\sum_{n=1}^{\infty} \sum_{k=k_n'+1}^{k_n-1} P\{A_{nk} \geqslant 1+\varepsilon\} \tag{1.1.36}$$

$$= c\sum_{n=1}^{\infty} \sum_{k=k_n'+1}^{k_n-1} \theta^{2\varepsilon'} (\log T_n)^{-(1+2\varepsilon')}$$

$$\leqslant c\sum_{n=1}^{\infty} k_n'' \theta^{2\varepsilon'} n^{-(1+2\varepsilon')}$$

$$\leqslant c\sum_{n=1}^{\infty} n^{-(1+\varepsilon')} < \infty .$$

Finally, merging (1.1.33), (1.1.35) and (1.1.36) together, we get

$$\sum_{n=1}^{\infty} P\{\sup_{-\infty<k\leqslant k_n-1} A_{nk} \geqslant 1+\varepsilon\} \leqslant \sum_{n=1}^{\infty} \sum_{-\infty<k\leqslant k_n-1} P\{A_{nk} \geqslant 1+\varepsilon\} < \infty$$

and (1.1.29) follows by the Borel-Cantelli lemma.

From (1.1.28) and (1.1.29), (1.1.25) holds true.

3° In order to finish the proof of (1.1.26), it is sufficient to show that

$$\varliminf_{T\to\infty} \sup_{0<t\leqslant T} \sup_{t\leqslant s\leqslant T} |W(s)-W(s-t)|/d(T,t) \geqslant 1 \quad \text{a.s.} \tag{1.1.37}$$

Let

$$B_n = \sup_{1\leqslant s\leqslant n} |W(s) - W(s-1)|/(2\log n)^{1/2}.$$

Using the well known probability inequality : for $x>0$

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right) \leqslant P\{W(1) \geqslant x\}$$

$$\leqslant \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \tag{1.1.38}$$

it follows that

$$\sum_{n=2}^{\infty} P\{B_n \leqslant 1-\varepsilon\}$$
$$\leqslant \sum_{n=2}^{\infty} P\{\max_{1\leqslant i\leqslant n} |W(i)-W(i-1)| \leqslant (1-\varepsilon)(2\log n)^{1/2}\}$$
$$= \sum_{n=2}^{\infty}\{1-\frac{c}{(\log n)^{1/2}}(\frac{1}{n})^{(1-\varepsilon)^2}\}^n$$
$$\leqslant \sum_{n=2}^{\infty} \exp\{-\frac{cn}{(\log n)^{1/2}}(\frac{1}{n})^{(1-\varepsilon)^2}\} < \infty,$$

so we have $\underline{\lim}_{n\to\infty} B_n \geqslant 1$ a.s. by the Borel-Cantelli lemma. Notice that when $n \leqslant T \leqslant n+1$, we have

$$\sup_{0<t\leqslant T}\sup_{t\leqslant s\leqslant T} |W(s)-W(s-t)|/d(T,t)$$
$$\geqslant \sup_{1\leqslant s\leqslant T} |W(s)-W(s-1)|/(2\log T)^{1/2}$$
$$\geqslant B_n[(\log n)/\log(n+1)]^{1/2},$$

therefore, conclusion (1.1.37) is proved.

Noting that the left-hand side of (1. 1. 25) $\leqslant$ the left-hand side of (1. 1. 27) $\leqslant$ the left-hand side of (1. 1. 26), we see that (1. 1. 27) is true from (1. 1. 25) and (1. 1. 26). The proof of Theorem 1. 1. 3 is now completed.

Chen, Kong and Lin (1986) pointed out that Theorem 1. 1. 3 can be reformulated in a general form, i.e. :

**Theorem 1.1.3$'$** *For $a_T$ such that $0<a_T\leqslant T$, we have*

$$\overline{\lim_{T\to\infty}} \sup_{a_T\leqslant t\leqslant T} |W(T)-W(T-t)|/d(T,t)=1 \quad \text{a.s.} \tag{1.1.25$'$}$$

$$\overline{\lim_{T\to\infty}} \sup_{a_T\leqslant t\leqslant T}\sup_{t\leqslant s\leqslant T} |W(s)-W(s-t)|/d(T,t)=1 \quad \text{a.s.} \tag{1.1.26$'$}$$

$$\overline{\lim_{T\to\infty}} \sup_{a_T\leqslant t\leqslant T}\sup_{t\leqslant s\leqslant T}\sup_{0\leqslant h\leqslant t} |W(s)-W(s-h)|/d(T,t)=1 \quad \text{a.s.} \tag{1.1.26$''$}$$

((1. 1. 26$''$), which was not mentioned by Chen, Kong and Lin (1986), can be proved emulating of (1.1.26$'$).)

**Corollary 1.1.1** (Hanson, Russo 1983a)

$$\lim_{a\to\infty} \sup_{\substack{0\leqslant u\leqslant s\leqslant t\leqslant v\\ a\leqslant v-u}} |W(t)-W(s)|/d(v,v-u)=1 \quad \text{a.s.} \tag{1.1.39}$$

*Proof* Note that

$$\overline{\lim_{a\to\infty}} \sup_{0\leqslant u\leqslant s\leqslant t\leqslant v,\, a\leqslant v-u} |W(t)-W(s)|/d(v,v-u)$$

$$=\lim_{a_0\to\infty} \sup_{0\leqslant u\leqslant s\leqslant t\leqslant v,\, a_0\leqslant a\leqslant v-u} |W(t)-W(s)|/d(v,v-u)$$

$$=\lim_{a\to\infty} \sup_{0\leqslant u\leqslant s\leqslant t\leqslant v,\, a\leqslant v-u} |W(t)-W(s)|/d(v,v-u). \tag{1.1.40}$$

Hence the limit (possibly infinite) in (1.1.39) exists.

Suppose $\omega$ is such that the "$\overline{\lim}$" in (1.1.27) is equal to one. Fix $\omega$. Choose $T_0$ large enough so that for given $\varepsilon>0$

$$\sup_{T_0\leqslant T} \sup_{0<t\leqslant T} \sup_{0\leqslant s\leqslant t} |W(T)-W(T-s)|/d(T,t)\leqslant 1+\varepsilon. \tag{1.1.41}$$

Putting $\tau=\min(v-u,t)$, we have

$$\sup_{0\leqslant u\leqslant s\leqslant t\leqslant v,\, a_0\leqslant a\leqslant v-u,\, T_0\leqslant t} |W(t)-W(s)|/d(v,v-u) \tag{1.1.42}$$

$$\leqslant \sup_{T_0\leqslant t} \sup_{a_0\leqslant v-u} \sup_{0\leqslant t-s\leqslant v-u,\, 0\leqslant s} |W(t)-W(s)|/d(v,v-u)$$

$$\leqslant \sup_{T_0\leqslant t} \sup_{0<\tau\leqslant t} \sup_{0\leqslant t-s\leqslant\tau} |W(t)-W(t-(t-s))|/d(t,\tau)\leqslant 1+\varepsilon.$$

In addition $\sup_{0\leqslant s\leqslant t\leqslant T_0} |W(t)-W(s)|$ is finite, and $d(v,v-u)\to\infty$ uniformly in $v$ as $v-u\to\infty$, so that

$$\overline{\lim_{a\to\infty}} \sup_{0\leqslant u\leqslant s\leqslant t\leqslant v,\, a\leqslant v-u,\, t\leqslant T_0} |W(t)-W(s)|/d(v,v-u)=0 \quad \text{a.s.} \tag{1.1.43}$$

Combining (1.1.42) with (1.1.43) and noting that $\varepsilon>0$ is arbitrary, we obtain (1.1.40) $\leqslant 1$ a.s. Letting $u=s=0$ and $t=v=a$ and using the law of the iterated logarithm, we have a contrary inequality. The conclusion of the corollary is proved.

*Remark* 1.1.1 Corollary 1.1.1 implies the following results (Hanson, Russo 1983 a): Suppose $0 < a_T \leqslant T$ and $a_T \to \infty$. Then

$$\overline{\lim_{T \to \infty}} \sup_{0 \leqslant u < v \leqslant T,\, a_T \leqslant v-u} |W(v) - W(u)|/d(v, v-u) = 1 \quad \text{a.s.} \qquad (1.1.44)$$

$$\overline{\lim_{T \to \infty}} \sup_{0 \leqslant u \leqslant s \leqslant t \leqslant v \leqslant T,\, a_T \leqslant v-u} |W(t) - W(s)|/d(v, v-u) = 1 \quad \text{a.s.} \qquad (1.1.45)$$

### 1.1.3 The general form of the increments

We will give a general form of the increments of a Wiener process. Both Csörgő-Révész's increments and a class of the lag increments are special cases of this general form of increments. We succeed in removing Conditions (i) and (ii) of Theorem 1.1.1. This general form was first discussed by Shao (1989).

**Theorem 1.1.4** *Let $a_T$, $b_T$ and $c_T$ be non-negative functions with $a_T + b_T \geqslant c_T \to \infty$ as $T \to \infty$. If there exists a constant $A > 0$ such that for any $T > 1$,*

$$b_T - b_{T-1} \leqslant A a_T, \; a_T + b_T \leqslant A(a_{T-1} + b_{T-1}), \qquad (1.1.46)$$

*then*

$$\overline{\lim_{T \to \infty}} \sup_{0 \leqslant t} \sup_{0 < s} \sup_{0 \leqslant r \leqslant s} |W(t+r) - W(t)|/d(t + s \vee c_T, s) = 1 \quad \text{a.s.} \qquad (1.1.47)$$

$$\overline{\lim_{T \to \infty}} \beta(a_T + b_T, a_T)|W(a_T + b_T) - W(b_T)| = 1 \quad \text{a.s.} \qquad (1.1.48)$$

*where*

$$\beta(M, m) = \{2m(\log M/m + \log\log M)\}^{-1/2}.$$

*Furthermore, if for any $0 < \varepsilon < 1$*

$$\sum_{N=1}^{\infty} \exp\{-b_N/a_N^{\varepsilon}((a_N + b_N)\log(a_N + b_N))^{1-\varepsilon}\} < \infty, \qquad (1.1.49)$$

*and*

$$\lim_{T \to \infty} b_T/b_{[T]} = \lim_{T \to \infty} a_T/a_{[T]} = 1, \qquad (1.1.50)$$

*then*

$$\lim_{T \to \infty} \sup_{0 \leqslant t} \sup_{0 < s} \sup_{0 \leqslant r \leqslant s} |W(t+r) - W(t)|/d(t + s \vee c_T, s) = 1 \quad \text{a.s.} \qquad (1.1.51)$$

$$\lim_{T \to \infty} \sup_{0 \leqslant t \leqslant b_T} \beta(t + a_T, a_T)|W(t + a_T) - W(t)| = 1 \quad \text{a.s.} \qquad (1.1.52)$$

*Proof* 1° We prove

$$\overline{\lim_{T\to\infty}} \sup_{0\leq t} \sup_{0<s} \sup_{0\leq r\leq s} |W(t+r)-W(t)|/d(t+s\vee c_T, s)\leq 1 \quad \text{a.s.} \tag{1.1.53}$$

Obviously, we can assume that $c_T\to\infty$ non-decreasingly as $T\to\infty$, otherwise we consider $c_T^*=\inf_{T\leq t} c_t$ instead of $c_T$.

For any given $B>0$, we have

$$\begin{aligned}
&\sup_{0\leq t} \sup_{0<s} \sup_{0\leq r\leq s} |W(t+r)-W(t)|/d(t+s\vee c_T, s)\\
&=(\sup_{0\leq t} \sup_{B<s} \sup_{0\leq r\leq s} |W(t+r)-W(t)|/d(t+s\vee c_T, s))\\
&\quad\vee(\sup_{0\leq t} \sup_{0<s\leq B} \sup_{0\leq r\leq s} |W(t+r)-W(t)|/d(t+s\vee c_T, s))\\
&=: I_1+I_2.
\end{aligned}$$

By Corollary 1.1.1, for any given $\varepsilon>0$ there exists a large $B=B(\varepsilon)$ such that $I_1\leq 1+\varepsilon$ a.s. Let $\theta>1$. Using (1.1.31), for large $T$ we have

$$\begin{aligned}
&P\{I_2\geq 1+\varepsilon\} \qquad\qquad (1.1.54)\\
&\leq\sum_{i=-\infty}^{\infty}\sum_{j=0}^{\infty} P\{\sup_{\theta^i\leq t<\theta^{i+1}} \sup_{B\theta^{-(j+1)}<s\leq B\theta^{-j}} \sup_{0\leq r\leq s} |W(t+r)-W(t)|/\sqrt{s}\\
&\geq(1+\varepsilon)(2(\log\frac{\theta^i+c_T}{B\theta^{-j}}+\log\log B\theta^{-j-1}))^{1/2}\}\\
&\leq c\sum_{i=-\infty}^{\infty}\sum_{j=0}^{\infty}\theta^i(\theta^i+c_T)^{-(1+\varepsilon)}\theta^{-j\varepsilon/2}\leq c\cdot c_T^{-\varepsilon}\to 0 \quad \text{as } T\to\infty.
\end{aligned}$$

Since $I_2=I_2(T)$ is a non-increasing function of $T$ as $T\to\infty$, $\overline{\lim}_{T\to\infty} I_2\leq 1+\varepsilon$ a.s. Thus we obtain (1.1.53).

2° In order to prove (1.1.47) and (1.1.48), we need only to show that for any $0<\varepsilon<1/8$

$$\overline{\lim_{N\to\infty}} \beta(a_N+b_N, a_N)|W(a_N+b_N)-W(b_N)|\geq 1-2\varepsilon \quad \text{a.s.} \tag{1.1.55}$$

Denote $N_1=1$;

$$N_{k+1}=\min\{n: n>N_k, b_n+\varepsilon^2 a_n\geq b_{N_k}+a_{N_k}\} \quad k\geq 1,$$

that is to say for every $k\geq 1$ and $n<N_{k+1}$, we have

$$b_{N_{k+1}}+\varepsilon^2 a_{N_{k+1}}\geqslant b_{N_k}+a_{N_k},\ b_n+\varepsilon^2 a_n<b_{N_k}+a_{N_k}. \tag{1.1.56}$$

It is easy to see that : $N_{k+1}>N_k$, $b_{N_{k+1}}+a_{N_{k+1}}>b_{N_k}+a_{N_k}$, $k\geqslant 1$.

Let us show that

$$\sum_{k=1}^{\infty} a_{N_k}/(b_{N_k}+a_{N_k})\log(b_{N_k}+a_{N_k})=\infty . \tag{1.1.57}$$

From (1.1.56) and (1.1.46), we have

$$b_{N_k-1}+a_{N_k-1}\geqslant b_{N_k-1}\geqslant b_{N_k}-Aa_{N_k}=b_{N_k}+a_{N_k}-(A+1)a_{N_k}, \tag{1.1.58}$$

$$b_{N_k-1}+a_{N_k-1}\geqslant \varepsilon^2(b_{N_k-1}+a_{N_k-1})\geqslant \varepsilon^2(b_{N_k}+a_{N_k})/A . \tag{1.1.59}$$

Then using (1.1.58) and (1.1.59), we obtain

$$\sum_{k=2}^{\infty}\frac{a_{N_k}}{(b_{N_k}+a_{N_k})\log(b_{N_k}+a_{N_k})}\geqslant\frac{1}{A+1}\sum_{k=2}^{\infty}\frac{b_{N_k}+a_{N_k}-(b_{N_{k-1}}+a_{N_{k-1}})}{(b_{N_k}+a_{N_k})\log(b_{N_k}+a_{N_k})}$$

$$\geqslant\frac{\varepsilon^2}{A(A+1)}\sum_{k=2}^{\infty}\frac{b_{N_k}+a_{N_k}-(b_{N_{k-1}}+a_{N_{k-1}})}{(b_{N_{k-1}}+a_{N_{k-1}})\log(A(b_{N_{k-1}}+a_{N_{k-1}})/\varepsilon^2)}$$

$$\geqslant\frac{\varepsilon^2}{A(A+1)}\sum_{k=2}^{\infty}\int_{b_{N_{k-1}}+a_{N_{k-1}}}^{b_{N_k}+a_{N_k}}\frac{1}{x\log(xA/\varepsilon^2)}dx=\infty .$$

Thus conclusion (1.1.57) holds true.

Denote $G=\{k: b_{N_k}\geqslant b_{N_{k-1}}+a_{N_{k-1}}\}$, $H=\{k: b_{N_k}<b_{N_{k-1}}+a_{N_{k-1}}\}$.

*Case* 1 Suppose that

$$\sum_{k\in G} a_{N_k}/(b_{N_k}+a_{N_k})\log(b_{N_k}+a_{N_k})=\infty . \tag{1.1.60}$$

Put

$$\beta(k)=\beta(b_{N_k}+a_{N_k},a_{N_k}),\ A_k=\{\beta(k)|W(b_{N_k}+a_{N_k})-W(b_{N_k})|\geqslant 1-\varepsilon\}.$$

Note that $A_k$, $k\in G$, are independent. Therefore in order to prove (1.1.55), we need only to show that

$$\sum_{k\in G}P(A_k)=\infty . \tag{1.1.61}$$

For large $k\in G$, we have

$$P(A_k)=P\{\beta(k)|W(a_{N_k})|\geqslant 1-\varepsilon\}$$

$$\geqslant c \exp\left\{ -(1-\varepsilon)\left(\log \frac{b_{N_k}+a_{N_k}}{a_{N_k}} + \log\log (b_{N_k}+a_{N_k})\right)\right\}$$
$$\geqslant c\, a_{N_k} /(b_{N_k}+a_{N_k})\log (b_{N_k}+a_{N_k}).$$

Hence (1.1.61) follows from assumption (1.1.60).

*Case* 2 Suppose that (1.1.60) fails. Then (1.1.57) implies

$$\sum_{k\in H} a_{N_k}/(b_{N_k}+a_{N_k})\log (b_{N_k}+a_{N_k})=\infty . \tag{1.1.62}$$

For any $k \in H$, from (1.1.56) we have

$$0\leqslant b_{N_{k-1}}+a_{N_{k-1}}-b_{N_k}\leqslant \varepsilon^2 a_{N_k} \tag{1.1.63}$$

and

$$(1-\varepsilon^2)a_{N_k}\leqslant (b_{N_k}+a_{N_k})-(b_{N_{k-1}}+a_{N_{k-1}})\leqslant a_{N_k} . \tag{1.1.64}$$

Note that

$$\beta(k)| W(b_{N_k}+a_{N_k})-W(b_{N_k})| \geqslant \tag{1.1.65}$$
$$\geqslant \beta(k)\{ | W(b_{N_k}+a_{N_k})-W(b_{N_{k-1}}+a_{N_{k-1}})|-| W(b_{N_{k-1}}+a_{N_{k-1}})-W(b_{N_k})|\}.$$

By similar arguments for (1.1.53), we have

$$\overline{\lim_{k\in H, k\to\infty}}\ \beta(k)|W(b_{N_{k-1}}+a_{N_{k-1}})-W(b_{N_k})| \leqslant\varepsilon \qquad \text{a.s.} \tag{1.1.66}$$

Now, in order to prove (1.1.55), we need only to show that

$$\overline{\lim_{k\in H, k\to\infty}}\ \beta(k)| W(b_{N_k}+a_{N_k})-W(b_{N_{k-1}}+a_{N_{k-1}})|\geqslant 1-\varepsilon \qquad \text{a.s.} \tag{1.1.67}$$

which can be obtained by emulating the proof of Case 1. Thus (1.1.47) and (1.1.48) hold true.

3° Finally, suppose that conditions (1.1.49) and (1.1.50) are satisfied. We show that

$$P\{ \max_{0\leqslant j\leqslant [b_N/a_N]} \beta(b_N+a_N , a_N)| W((j+1)a_N)-W(ja_N)|\leqslant 1-\varepsilon \quad \text{i.o.}\}=0 \tag{1.1.68}$$

Since $|W((j+1)a_N)-W(ja_N)|$, $j=0, 1, \cdots, [b_N/a_N]$, are independent, using (1.1.38) we get

$$P\{ \max_{0\leqslant j\leqslant [b_N/a_N]} \beta(b_N+a_N , a_N)| W((j+1)a_N)-W(ja_N)|\leqslant 1-\varepsilon \}$$

$$\leqslant (1-(\frac{a_N}{(b_N+a_N)\log(b_N+a_N)})^{1-\varepsilon})^{b_N/a_N}$$

$$\leqslant \exp\{-b_N/a_N^{\varepsilon}((b_N+a_N)\log(b_N+a_N))^{1-\varepsilon}\}.$$

By condition (1.1.49), it follows that (1.1.68) holds true, that is to say, we have

$$\varliminf_{N\to\infty} \max_{0\leqslant j\leqslant b_N/a_N} \beta(b_N+a_N, a_N)|W((j+1)a_N)-W(ja_N)| \geqslant 1-\varepsilon \quad \text{a.s.}$$

Since $a_T/a_{[T]} \to 1$, $b_T/b_{[T]} \to 1$, then we have

$$\varliminf_{T\to\infty} \max_{0\leqslant j\leqslant [b_T/a_{[T]}]} \beta(b_T+a_T, a_T)|W((j+1)a_{[T]})-W(ja_{[T]})| \geqslant 1-\varepsilon \quad \text{a.s.} \tag{1.1.69}$$

Note that we have the inequality

$$\varliminf_{T\to\infty} \sup_{0\leqslant t\leqslant b_T} \beta(t+a_T, a_T)|W(t+a_T)-W(t)| \tag{1.1.70}$$

$$\geqslant \varliminf_{T\to\infty} \sup_{0\leqslant j\leqslant [b_T/a_{[T]}]} \beta(b_T+a_T, a_T)|W((j+1)a_{[T]})-W(ja_{[T]})|$$

$$-\varlimsup_{T\to\infty} \sup_{0\leqslant t\leqslant b_T} \sup_{0\leqslant s\leqslant |a_T-a[T]|} \beta(b_T+a_T, a_T)|W(t+s)-W(t)|.$$

The second term of the right-hand side does not exceed $\varepsilon$. So merging (1.1.69), (1.1.70) together, we get

$$\varliminf_{T\to\infty} \sup_{0\leqslant t\leqslant b_T} \beta(t+a_T, a_T)|W(t+a_T)-W(t)| \geqslant 1-2\varepsilon \quad \text{a.s.}$$

Thus (1.1.51) and (1.1.52) hold true. The proof of Theorem 1.1.4 is completed.

*Remark* 1.1.2 Let $0<a_T\leqslant T$. From (1.1.47) we have

$$\varlimsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} \beta_T |W(t+s)-W(t)| \tag{1.1.47$'$}$$

$$\leqslant \varlimsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} |W(t+s)-W(t)|/d(t+a_T, a_T) \leqslant 1 \quad \text{a.s.}$$

On the other hand, taking $b_T=T$ in (1.1.48), we have

$$\varlimsup_{T\to\infty} \beta_T|W(T+a_T)-W(T)| = \varlimsup_{T\to\infty} \beta(T+a_T, a_T)|W(T+a_T)-W(T)| = 1 \quad \text{a.s.} \tag{1.1.48$'$}$$

provided that

$$\underline{\lim}_{T \to \infty} a_T > 0 . \tag{1.1.71}$$

Combining (1.1.47 ′) with (1.1.48 ′) implies (1.1.1)—(1.1.4) of Theorem 1.1.1, that is to say Conditions (i) and (ii) can be replaced by Condition (1.1.71) which is trivial.

It is easy to see that (1.1.47) implies the following results of the lag increments (Hanson, Russo 1983a and Chen, Kong, Lin 1986). If $a_T \to \infty$

$$\overline{\lim}_{T \to \infty} \sup_{0 \leqslant t \leqslant T - a_T} |W(t + a_T) - W(t)| / d(t + a_T, a_T) \leqslant 1 \quad \text{a.s.} \tag{1.1.72}$$

$$\overline{\lim}_{T \to \infty} \sup_{0 \leqslant t \leqslant T - a_T} \sup_{0 \leqslant s \leqslant a_T} |W(t + s) - W(t)| / d(t + a_T, a_T) \leqslant 1 \quad \text{a.s.} \tag{1.1.73}$$

If, in addition, $a_T$ is onto, then we have equality in (1.1.72) and (1.1.73). Furthermore, if $a_T$ is a continuous function of $T$ and satisfies Condition (iii) of Theorem 1.1.1, then $\overline{\lim}$ can be replaced by lim and can have equalities in (1.1.72) and (1.1.73).

If, in addition to the condition (1.1.71), we have

$$\sum_{N=1}^{\infty} \exp\left\{ -\left( \frac{N \log N}{a_N} \right)^{\varepsilon} / \log N \right\} < \infty \qquad \text{for any } \varepsilon > 0 \tag{1.1.49 ′}$$

and

$$\lim_{T \to \infty} a_T / a_{[T]} = 1 , \tag{1.1.50 ′}$$

then (1.1.5) and (1.1.6) of Theorem 1.1.1 are also true.

## 1.2 Some Inferior Limit Results for the Increments of a Wiener Process

In Section 1.1, we discussed the increments of a Wiener process and obtained some superior limit results. Some inferior limit results for Csörgő-Révész's increments have also been mentioned. In this section, we will investigate the details of the inferior limit results for the lag increments.

He and Chen (1989) investigated the inferior limit version of Theorem 1.1.3$'$ and obtained the following theorem which corresponds to the result of Book and Shore (1978).

**Theorem 1.2.1** (He, Chen 1989) *Let $0 < a_T \leqslant T$ be a non-decreasing function of $T$ and satisfy*

$$\text{(iv)} \qquad \lim_{T\to\infty} (\log T/a_T)/\log\log T = r \qquad 0 \leqslant r \leqslant \infty .$$

*Then we have*

$$\varliminf_{T\to\infty} \sup_{a_T \leqslant t \leqslant T} \sup_{t \leqslant s \leqslant T} |W(s) - W(s-t)|/d(T,t) = \alpha_r \qquad \text{a.s.} \qquad (1.2.1)$$

$$\varliminf_{T\to\infty} \sup_{a_T \leqslant t \leqslant T} \sup_{t \leqslant s \leqslant T} \sup_{0 \leqslant h \leqslant t} |W(s) - W(s-h)|/d(T,t) = \alpha_r \qquad \text{a.s.} \quad (1.2.2)$$

*where* $\alpha_r = (r/(r+1))^{1/2}$.

*Proof* First, we prove that

$$\varliminf_{T\to\infty} \sup_{a_T \leqslant t \leqslant T} \sup_{t \leqslant s \leqslant T} |W(s) - W(s-t)|/d(T,t) \geqslant \alpha_r \qquad \text{a.s.} \qquad (1.2.3)$$

It is clear that (1.2.3) is true for the case of $r=0$. For the case of $0 < r \leqslant \infty$, it follows from (iv) that Condition (vi) is satisfied. Hence by Theorem 1.1.2 we have the left-hand side of (1.2.3)

$$\geqslant \lim_{T\to\infty} \sup_{0 \leqslant s \leqslant T-a_T} |W(s+a_T) - W(s)|/d(T,a_T)$$

$$\geqslant \lim_{T\to\infty} \left( \frac{2a_T(\log T/a_T - \log\log\log T)}{2a_T(\log T/a_T + \log\log a_T)} \right)^{1/2} \geqslant \alpha_r \qquad \text{a.s.}$$

Now, in order to prove Theorem 1.2.1, it suffices to show that

$$\varliminf_{T\to\infty} \sup_{a_T \leqslant t \leqslant T} \sup_{t \leqslant s \leqslant T} \sup_{0 \leqslant h \leqslant t} |W(s) - W(s-h)|/d(T,t) \leqslant \alpha_r \qquad \text{a.s.} \quad (1.2.4)$$

It is clear by (1.1.26$''$) that (1.2.4) holds true for the case of $r=\infty$. We need only to prove that

$$\varlimsup_{n\to\infty} \sup_{a_{T_n} \leqslant t \leqslant T_n} \sup_{t \leqslant s \leqslant T_n} \sup_{0 \leqslant h \leqslant t} |W(s) - W(s-h)|/d(T_n, t) \leqslant \alpha_r \qquad \text{a.s.} \quad (1.2.5)$$

for $T_n = e^{e^n}$ and $0 \leqslant r < \infty$. For any $\alpha_r < \alpha < 1$, we take $\theta > 1, \varepsilon > 0$ such that

$$2\alpha^2 /(2+\varepsilon)\theta > r/(r+1) + \varepsilon .$$

Put $K_n = [\log_\theta T_n / a_{T_n}]$, $t_k = \theta^k a_{T_n}$, $0 \leqslant k \leqslant K_n$. We have

$$\begin{aligned}
&\sup_{a_{T_n} \leqslant t \leqslant T_n} \sup_{t \leqslant s \leqslant T_n} \sup_{0 \leqslant h \leqslant t} |W(s) - W(s-h)| / d(T_n, t) \\
&\leqslant \max_{0 \leqslant k \leqslant K_n} \sup_{t_k \leqslant t \leqslant t_{k+1}} \sup_{t \leqslant s \leqslant T_n} \sup_{0 \leqslant h \leqslant t} |W(s) - W(s-h)| / d(T_n, t_k) \\
&\leqslant \max_{0 \leqslant k \leqslant K_n} \sup_{0 \leqslant s-h, s \leqslant T_n} \sup_{0 \leqslant h \leqslant t_{k+1}} |W(s) - W(s-h)| / d(T_n, t_k) \\
&=: \max_{0 \leqslant k \leqslant K_n} A_{nk} .
\end{aligned}$$

It follows from Condition (iv) and (1.1.31) that, for large $n$, we have

$$\begin{aligned}
P(A_{nk} \geqslant \alpha) &\leqslant c \frac{T_n}{t_{k+1}} \exp\left\{ -\frac{2\alpha^2}{(2+\varepsilon)\theta} \left(\log \frac{T_n}{t_k} + \log \log t_k\right)\right\} \\
&\leqslant c (\log T_n)^{(r+\varepsilon) - (r+\varepsilon+1)\frac{2\alpha^2}{(2+\varepsilon)\theta}} \cdot \theta^{-k(1-\alpha^2)} \\
&\leqslant c\, e^{-n\varepsilon^2} \theta^{-k(1-\alpha^2)}.
\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} P\{ \max_{0 \leqslant k \leqslant K_n} A_{nk} \geqslant \alpha \} < \infty .$$

By the Borel-Cantelli lemma (1.2.5) holds true. The proof is completed.

Hanson and Russo (1989) have also investigated the inferior limits for other sorts of lag increments of a Wiener process.

**Theorem 1.2.2** (Hanson, Russo 1989) *Let* $0 < a_T \leqslant T$. *Then*

$$\varliminf_{T \to \infty} \sup_{0 < t \leqslant T} \sup_{0 \leqslant s \leqslant t} |W(T) - W(T-s)| / d(T, t) = 0 \quad \text{a.s.} \tag{1.2.6}$$

$$\varliminf_{T \to \infty} \sup_{a_T \leqslant t \leqslant T} (W(T) - W(T-t)) / d(T, t) \begin{cases} \text{is in } [-1, g(a)] & \text{if } \varliminf_{T \to \infty} a_T / T = a, \\ = g(a) & \text{if } \lim_{T \to \infty} a_T / T = a, \end{cases}$$

*where*

$$g(a)=\begin{cases}0 & \text{if } a=0\,,\\ -1/(1-(\log a)/4)^{1/2} & \text{if } 0<a\leqslant 1.\end{cases} \tag{1.2.7}$$

We will use the following two lemmas in our proof.

**Lemma 1.2.1** (Strassen 1964) *Define*

$$\eta_T(x)=W(Tx)/(2T\log\log T)^{1/2}\qquad 0\leqslant x\leqslant 1.$$

*The sequence $\{\eta_T(x)\}$ is relatively compact in $C[0,1]$ with probability one, and the set of its limit points is $K$, where $K$ is the set of absolutely continuous functions (with respect to the Lebesgue measure) such that $f(0)=0$ and $\int_0^1 (f'(x))^2\, dx\leqslant 1$.*

The proof can be found in Csörgő and Révész (1981, Theorem 1.3.2).

**Lemma 1.2.2** *Suppose $a<b$ and $f(x)=\alpha x+\beta$ for $x=a$ and $x=b$. Suppose also that $f$ is absolutely continuous on $[a,b]$ with the Radon-Nikodym derivative $f'$. Then*

$$\int_a^b (f'(x))^2dx\geqslant\int_a^b \alpha^2dx$$

*and the equality holds if and only if $f(x)=\alpha x+\beta$ for all $x$ in $[a,b]$.*

*Proof* Let $\mu$ be Lebesgue measure, $P=\mu/(b-a)$. Then $X=f'$ is a random variable on the probability space $([a,b],\Sigma,P)$, where $\Sigma$ is the collection of $\mu$-measurable subsets of $[a,b]$.

$$\frac{1}{b-a}\int_a^b (f'(x))^2dx=EX^2=E(X-EX)^2+(EX)^2$$

$$=E(X-\alpha)^2+\alpha^2\geqslant\alpha^2=\frac{1}{b-a}\int_a^b\alpha^2dx$$

and the inequality is strict unless $f'=X=\alpha$. a.s.

*Proof of Theorem* 1.2.2.

1° Let us prove (1.2.6). Note that

$$\lim_{T \to \infty} \beta(T, t) \cdot d(T, t) = 1$$

uniformly for $0 < t \leqslant T$. So it needs only to prove (1.2.6) with $d(T,t)$ replaced by $\beta^{-1}(T, t)$.

Fix $\varepsilon > 0$. For $k \geqslant 1$ and $-\infty < n \leqslant k$, define

$$E_{nk} = \{ \sup_{0 \leqslant t \leqslant 2^n} | W(2^k) - W(2^k - t) | \beta(2^k, 2^n) > \sqrt{\varepsilon} \},$$

$$E_k = \bigcup_{-\infty < n \leqslant k} E_{nk}.$$

Then for all large $k$

$$P(E_{nk}) \leqslant 4P\{ W(2^n) > \sqrt{\varepsilon}\, \beta^{-1}(2^k, 2^n) \}$$
$$\leqslant e^{-\varepsilon (\log 2^{k-n} + \log \log 2^k)}$$
$$\leqslant c\, 2^{-\varepsilon(k-n)} k^{-\varepsilon},$$

which implies

$$P(E_k) \leqslant c 2^{-\varepsilon k} k^{-\varepsilon} \sum_{-\infty < n \leqslant k} (2^\varepsilon)^n = c k^{-\varepsilon} \to 0 \quad \text{as} \quad k \to \infty .$$

Hence there exists a subsequence $\{k'\}$ such that $P(\overline{\lim_{k'}}\, E_{k'}) = 0$. If $\omega \in (\overline{\lim}\, E_{k'})^c$, then $\omega \in E^c_{k'}$ for all large $k'$, say $k' \geqslant K$. Suppose $k' \geqslant K$ and $0 < t \leqslant 2^{k'}$, so that $2^{n-1} \leqslant t \leqslant 2^n$ for some $-\infty < n \leqslant k'$. Then if $0 \leqslant s \leqslant t$, since $\omega$ is not in $E_{nk'}$, we have

$$| W(2^{k'}) - W(2^{k'} - s) | \beta(2^{k'}, t)$$
$$\leqslant \sup_{0 \leqslant s \leqslant 2^n} | W(2^k) - W(2^k - s) | \beta(2^k, 2^n) \sqrt{2} \leqslant \sqrt{2\varepsilon} .$$

This proves (1.2.6).

2° We prove that if $a_T/T \to a > 0$, then

$$\varlimsup_{T \to \infty} \sup_{a_T \leqslant t \leqslant T} (W(T) - W(T-t))/d(T,t) = g(a) \qquad \text{a.s.} \quad (1.2.8)$$

Since $a_T/T \to a > 0$, for $T$ large enough, $\log T/t$ is bounded uniformly in $t$ for $a_T \leqslant t \leqslant T$. Thus we can replace $d(T, t)$ by $(2t \log \log t)^{1/2}$. By Lemma 1.2.1, there is a $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, such that for $\omega \in \Omega_0$, $\{\eta_T(x, \omega)\}$ is relatively compact in $C[0, 1]$. Fix $\omega_0$ in the $\Omega_0$. Suppose that $\{T_n\}$ is chosen so that $T_n \to \infty$ and for $\omega_0$

$$\lim_{n \to \infty} \sup_{a_{T_n} \leqslant t \leqslant T_n} \frac{W(T_n) - W(T_n - t)}{(2t \log \log t)^{1/2}} = \varlimsup_{T \to \infty} \sup_{a_T \leqslant t \leqslant T} \frac{W(T) - W(T-t)}{d(T, t)} .$$

From Lemma 1.2.1, there exists a subsequence of $T_n$ (which we will still call $T_n$) and an $f_0$ in $K$ such that

$$W(T_n x)/\sqrt{2T_n \log\log T_n} - f_0(x) \to 0$$

uniformly for $x$ in $[0, 1]$. Thus

$$\frac{W(T_n \cdot 1) - W(T_n(1 - t/T_n))}{\sqrt{2T_n \log\log T_n}} - (f_0(1) - f_0(1 - t/T_n)) \to 0$$

uniformly for $0 \leqslant t \leqslant T_n$ as $n \to \infty$. It follows that

$$\varliminf_{T \to \infty} \sup_{a_T \leqslant t \leqslant T} (W(T) - W(T-t))/d(T, t) \tag{1.2.9}$$

$$= \varliminf_{n \to \infty} \sup_{a_{T_n} \leqslant t \leqslant T_n} \frac{W(T_n) - W(T_n - t)}{\sqrt{2T_n \log\log T_n}} \sqrt{\frac{T_n}{t}}$$

$$= \sup_{a \leqslant s \leqslant 1} \frac{f_0(1) - f_0(1-s)}{\sqrt{s}}$$

$$\geqslant \inf_{f \in K} \sup_{a \leqslant s \leqslant 1} \frac{f(1) - f(1-s)}{\sqrt{s}} .$$

Using the same approximation argument, it follows from Lemma 1. 2. 1 that if $f^*$ is in $K$, then there is a subsequence $T_n \to \infty$ such that

$$\lim_{n \to \infty} \sup_{a_{T_n} \leqslant t \leqslant T_n} \frac{W(T_n) - W(T_n - t)}{d(T_n, t)} = \sup_{a \leqslant s \leqslant 1} \frac{f^*(1) - f^*(1-s)}{\sqrt{s}} .$$

Since $K$ is compact, there is an $f^*$ in $K$ such that

$$\sup_{a \leqslant s \leqslant 1} \frac{f^*(1) - f^*(1-s)}{\sqrt{s}} = \inf_{f \in K} \sup_{a \leqslant s \leqslant 1} \frac{f(1) - f(1-s)}{\sqrt{s}} .$$

This depends on $a$, but not otherwise on the sequence $a_T$, and is our function $g(a)$ whose functional form we have yet to determine. By symmetry

$$g(a) = - \sup_{f \in K} \inf_{a \leqslant s \leqslant 1} (f(1) - f(1-s))/\sqrt{s}$$

$$= - \sup_{f \in K} \inf_{a \leqslant s \leqslant 1} f(s)/\sqrt{s} .$$

Suppose $h$ is a function in $K$ which achieves the sup, i.e.

$$-g(a) = \sup_{f \in K} \inf_{a \leqslant s \leqslant 1} f(s)/\sqrt{s} = \inf_{a \leqslant s \leqslant 1} h(s)/\sqrt{s} . \tag{1.2.10}$$

Taking $f(s) = s$, we get $-g(a) > 0$. From (1.2.10), we have

$$h(s)/\sqrt{s} \geqslant -g(a) \tag{1.2.11}$$

for $a \leqslant s \leqslant 1$, and there is equality for at least one $s$ in $[a, 1]$.

Now we prove that

$$h(s) \begin{cases} \text{is linear} & \text{if } 0 \leqslant s \leqslant a , \\ = -g(a)\sqrt{s} & \text{if } a \leqslant s \leqslant 1. \end{cases} \tag{1.2.12}$$

Otherwise, suppose that $h(s)$ is not linear on $[0, a]$. Define

$$h_1(s) = \begin{cases} sh(a)/a & \text{if } 0 \leqslant s \leqslant a, \\ h(s) & \text{if } a \leqslant s \leqslant 1. \end{cases}$$

Then $\inf_{a \leqslant s \leqslant 1} h_1(s)/\sqrt{s} = \inf_{a \leqslant s \leqslant 1} h(s)/\sqrt{s}$ and, from Lemma 1.2.2,

$$\int_0^1 (h_1'(s))^2\,ds < \int_0^1 (h'(s))^2 ds \leqslant 1.$$

The function $h_2(s) = h_1(s)/(\int_0^1 (h_1'(s))^2 ds)^{1/2}$ belongs to $K$, but $\inf_{a \leqslant s \leqslant 1} h_2(s)/\sqrt{s} > -g(a)$ leads to a contradiction to (1.2.10). Next, suppose that there exists an $s^* \in [a, 1]$ such that $h(s^*) \neq -g(a)\sqrt{s^*}$. By (1.2.11), $h(s^*) > -g(a)\sqrt{s^*}$. Let $h_1(s)$ be the tangent to $y = -g(a)\sqrt{s}$ at $s = s^*$ and let $h_2 = \min\{h, h_1\}$. The function $h_2$ would satisfy $h_2(s)/\sqrt{s} \geqslant -g(a)$ for all $s$ in $[a, 1]$. By Lemma 1.2.2 again, $\int_0^1 (h_2'(s))^2 ds < \int_0^1 (h'(s))^2 ds \leqslant 1$.

Then, the function $h_3(s) = h_2(s)/\{\int_0^1 (h_2'(s))^2 ds\}^{1/2}$ would be in $K$, and $\inf_{a \leqslant s \leqslant 1} h_3(s)/\sqrt{s} > -g(a)$ would give a contradiction.

By definition, we have

$$\int_0^1 (h'(s))^2 ds = 1 \tag{1.2.13}$$

(otherwise we can multiply $h(s)$ by a constant) and from (1.2.12) we have

$$h(s) = \begin{cases} -g(a)s/\sqrt{a} & \text{if } 0 \leqslant s \leqslant a, \\ -g(a)\sqrt{s} & \text{if } a \leqslant s \leqslant 1. \end{cases}$$

So that

$$h'(s) = \begin{cases} -g(a)/\sqrt{a} & \text{if } 0 \leqslant s \leqslant a, \\ -g(a)/(2\sqrt{s}) & \text{if } a \leqslant s \leqslant 1. \end{cases} \tag{1.2.14}$$

From (1.2.13) and (1.2.14), we get

$$g(a) = -(1 - (\log a)/4)^{-1/2}.$$

3° If $a_T/T \to 0$, then

$$\varliminf_{T\to\infty} \sup_{a_T\leq t\leq T} \frac{W(T)-W(T-t)}{d(T,t)} \geq \varliminf_{T\to\infty} \sup_{a_T\leq t\leq T} \frac{W(T)-W(T-t)}{d(T,t)}$$
$$=g(a) \quad \text{a.s.} \tag{1.2.15}$$

for every $a>0$, so (1.2.15) is bounded below by $0=g(0)$. On the other hand, it is bounded above by 0 from (1.2.6).

4° We always have

$$-1=g(1)=\varliminf_{T\to\infty} \sup_{1\cdot T\leq t\leq T} (W(T)-W(T-t))/d(T,t)$$
$$\leq \varliminf_{T\to\infty} \sup_{a_T\leq t\leq T} (W(T)-W(T-t))/d(T,t) \quad \text{a.s.} \tag{1.2.16}$$

If $\varliminf_{T\to\infty} a_T/T=a=0$, then, from (1.2.6) we see that (1.2.16) is bounded above by $0=g(0)$. If $\varliminf_{T\to\infty} a_T/T=a>0$, then for any $0<\varepsilon<a$ we have the right-hand side of (1.2.16)

$$\leq \varliminf_{T\to\infty} \sup_{(a-\varepsilon)T\leq t\leq T} (W(T)-W(T-t))/d(T,t)=g(a-\varepsilon)$$

and the last part of the proof of Theorem 1.2.2 is completed by letting $\varepsilon\to 0$.

*Remark* 1.2.1 Some other problems about the lag increments of a Wiener process, such as the set of the limit points of $\beta_T(W(T,\omega)-W(T-a_T,\omega))$ and $\sup_{a_T\leq t\leq T}|W(T,\omega)-W(T-t,\omega)|/d(T,t)$ have been investigated by Hanson and Russo (1983a, b), Chen, Kong and Lin (1986) and Liu (1986), etc.

## 1.3 Further Discussion for Increments of a Wiener Process

Since the problem of increments of a Wiener process appeared, the accurate rate of convergence of some increments have been investigated by various authors in the past decade.

### 1.3.1 The accurate rate of convergence of the increments of a Wiener process

Let $\{X(t);\ t\geq 0\}$ be a stochastic process on a probability space $(\Omega,\mathscr{F},P)$. We introduce the following definitions.

**Definition 1.3.1** *The function $a_1(t)$, $t \geqslant 0$, belongs to the upper-upper class of the process $X(t)$ ($a_1 \in UUC(X)$), if for almost all $\omega \in \Omega$ there exists a $t_0 = t_0(\omega)$ such that $X(t) < a_1(t)$ for every $t > t_0$.*

**Definition 1.3.2** *The function $a_2(t)$, $t \geqslant 0$, belongs to the upper-lower class of the process $X(t)$ ($a_2 \in ULC(X)$), if for almost all $\omega \in \Omega$ there exists a sequence $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \cdots$ with $t_i \to \infty$ ($i \to \infty$), such that $X(t_i) \geqslant a_2(t_i)$, $i = 1, 2, \cdots$.*

**Definition 1.3.3** *The function $a_3(t)$, $t \geqslant 0$, belongs to the lower-upper class of the process $X(t)$ ($a_3 \in LUC(X)$), if for almost all $\omega \in \Omega$ there exists a sequence $0 < t_1 = t_1(\omega) < t_2 = t_2(\omega) < \cdots$ with $t_i \to \infty$ ($i \to \infty$), such that $X(t_i) \leqslant a_3(t_i)$, $i = 1, 2, \cdots$.*

**Definition 1.3.4** *The function $a_4(t)$, $t \geqslant 0$, belongs to the lower-lower class of the process $X(t)$ ($a_4 \in LLC(X)$), if for almost all $\omega \in \Omega$ there exists a $t_0 = t_0(\omega)$ such that $X(t) > a_4(t)$ for every $t > t_0$.*

Let $0 < a_T \leqslant T$ be a non-decreasing function of $T$, denote

$$Y_1(T) = a_T^{-1/2} \sup_{0 \leqslant t \leqslant T - a_T} (W(t + a_T) - W(t)),$$

$$Y_2(T) = a_T^{-1/2} \sup_{0 \leqslant t \leqslant T - a_T} |W(t + a_T) - W(t)|,$$

$$Y_3(T) = a_T^{-1/2} \sup_{0 \leqslant t \leqslant T - a_T} \sup_{0 \leqslant s \leqslant a_T} (W(t + s) - W(t)),$$

$$Y_4(T) = a_T^{-1/2} \sup_{0 \leqslant t \leqslant T - a_T} \sup_{0 \leqslant s \leqslant a_T} |W(t + s) - W(t)|.$$

We have introduced how Large the increments of a Wiener process are in Section 1.1.1, furthermore, we intend to study the four classes of processes $Y_i(t)$, $i = 1, 2, 3, 4$, and to give more precise limit results for these four forms of the increments of a Wiener process.

**Theorem 1.3.1** (Révész 1982) *Let $0 < a_T \leqslant T$ be a function of $T$ for which*

(i) *$a_T$ is non-decreasing,*

(ii) *$T/a_T$ is non-decreasing,*

(iii) $\lim\limits_{T\to\infty}(\log T/a_T)/\log\log T=\infty$ ,

*and put*

$$a_1(T)=a_1(T,\varepsilon)=(2\log T/a_T+2\log\log T+(3+\varepsilon)\log\log T/a_T$$
$$+(2+\varepsilon)\log\log\log T)^{1/2},$$

$$a_2(T)=(2\log T/a_T+2\log\log T+\log\log T/a_T+2\log\log\log T)^{1/2},$$

$$a_3(T)=a_3(T,\varepsilon)=(2\log T/a_T+\log\log T/a_T-2\log\log\log T$$
$$+\log(\frac{51^2}{\pi}+\varepsilon))^{1/2},$$

$$a_4(T)=a_4(T,\varepsilon)=(2\log T/a_T+\log\log T/a_T-2\log\log\log T$$
$$-\log(\pi(1+\varepsilon)))^{1/2}.$$

*Then for any* $\varepsilon>0$ *and* $i=1,2,3,4$, *we have*

$$a_1(t)\in UUC(Y_i), \tag{1.3.1}$$

$$a_2(t)\in ULC(Y_i), \tag{1.3.2}$$

$$a_3(t)\in LUC(Y_i), \tag{1.3.3}$$

$$a_4(t)\in LLC(Y_i). \tag{1.3.4}$$

*Proof* 1° Proof of (1.3.1). Since

$$Y_1(T)=\min(Y_1(T),\ Y_2(T),\ Y_3(T),\ Y_4(T))$$
$$\leqslant\max(Y_1(T),\ Y_2(T),\ Y_3(T),\ Y_4(T))=Y_4(T),$$

it is enough to prove (1.3.1) for $i=4$. Denote

$$P(T,\varepsilon)=P\{Y_4(T)\geqslant a_1(T,\varepsilon)\}.$$

Then by Lemma 1.1.4, we have

$$P(T,\varepsilon/2)=O((\log T)^{-1}(\log T/a_T)^{-1-\varepsilon/4}(\log\log T)^{-1-\varepsilon/4})$$
$$=O((\log T)^{-1}(w(T))^{-1-\varepsilon/4}(\log\log T)^{-2-\varepsilon/2}), \tag{1.3.5}$$

where $w(T)=(\log T/a_T)/\log\log T$. Now let $T_k$ be the smallest real number for which

$$(\log T_k)(w(T_k))^{1+\varepsilon/4}(\log\log T_k)=k. \tag{1.3.6}$$

Then by the trivial inequality $\log w(T) < \log\log T$, we have

$$P(T_k, \varepsilon/2) = O(k^{-1}(\log\log T_k)^{-1-\varepsilon/2}) = O(k^{-1}(\log k)^{-1-\varepsilon/2}). \qquad (1.3.7)$$

By the Borel-Cantelli lemma, we have

$$P\{\, Y_4(T_k) \geqslant a_1(T_k, \varepsilon/2) \quad \text{i.o.}\,\} = 0.$$

Statement (1.3.1) follows from the fact that the process $a_T^{1/2} Y_4(T)$ is non-decreasing and from the inequality

$$a_{T_{k+1}}\, a_1(T_{k+1}, \varepsilon/2) \leqslant a_{T_k}\, a_1(T_k, \varepsilon).$$

In the proof of the last inequality the following trivial relations should be utilized:

$$\frac{a_{T_{k+1}}}{a_{T_k}} \leqslant \frac{T_{k+1}}{T_k} = O((\log\log T_k)^{-1}(w(T_k))^{-1-\varepsilon/4}) + 1,$$

$$\log\frac{T_{k+1}}{a_{T_{k+1}}} \leqslant \log\frac{T_k}{a_{T_k}} + \log\frac{T_{k+1}}{T_k}$$
$$= w(T_k)\log\log T_k + O((\log\log T_k)^{-1}(w(T_k))^{-1-\varepsilon/4}).$$

2° Proof of (1.3.2). It is enough to prove it for $i = 1$. In fact, the following stronger statement will be proved:

$$P\{A_k \text{ i.o.}\} := P\{\sup_{T_k \leqslant s \leqslant T_{k+1} - a_{T_{k+1}}} a_{T_{k+1}}^{-1/2}(W(s + a_{T_{k+1}}) - W(s))$$
$$\geqslant a_2(T_{k+1}) \quad \text{i.o.}\,\} = 1,$$

where $\{T_k\}$ is defined by (1.3.6). From (1.3.6) we have $T_{k+1} - T_k \geqslant a_{T_{k+1}}$. By Lemma 1.1.3 we have

$$P(A_k) = O\left(\frac{T_{k+1} - T_k}{a_{T_{k+1}}}\left(\log\frac{T_{k+1}}{a_{T_{k+1}}}\right)^{1/2}\frac{a_{T_{k+1}}}{T_{k+1}}(\log T_{k+1})^{-1}\right.$$
$$\left.\cdot\left(\log\frac{T_{k+1}}{a_{T_{k+1}}}\right)^{-1/2}(\log\log T_{k+1})^{-1}\right)$$
$$= O((\log\log T_{k+1})^{-2}(w(T_{k+1}))^{-1-\varepsilon/4}(\log T_{k+1})^{-1})$$
$$= O((k\log\log T_{k+1})^{-1}) = O((k\log k)^{-1}),$$

which proves (1.3.2).

3° Proof of (1.3.3). It is enough the prove this for $i = 4$. Let

$T_k = \exp(k^{1+\rho}),\ k=1,2,\cdots;\rho>0$, and let

$$Z_4(k+1) = \sup_{T_k \leqslant t \leqslant T_{k+1} - a_{T_{k+1}}} \sup_{0 \leqslant s \leqslant a_{T_{k+1}}} a_{T_{k+1}}^{-1/2} | W(t+s) - W(t)| .$$

Then by Lemma 1.1.4 we have

$$\sum_{k=1}^{\infty} P\{ Z_4(k) < a_3(T_k)\} = \infty .$$

This proves the statement

$$P\{ Z_4(k) < a_3(T_k) \ \text{ i.o. }\} = 1.$$

Since

$$Y_4(T_{k+1}) \leqslant Z_4(k+1) + \sup_{0 \leqslant t \leqslant T_k} \sup_{0 \leqslant s \leqslant a_{T_{k+1}}} a_{T_{k+1}}^{-1/2} | W(t+s) - W(t)| ,$$

and by (1.3.1)

$$\sup_{0 \leqslant t \leqslant T_k} \sup_{0 \leqslant s \leqslant a_{T_{k+1}}} | W(t+s) - W(t)| = o\,(a_{T_{k+1}}^{1/2}\, a_3(T_{k+1})),$$

we have (1.3.3).

4° Proof of (1.3.4). It is enough to prove this for $i=1$. By Lemma 1.1.3 we have

$$P\{ Y_1(T) \leqslant a_4(T, 3\varepsilon)\} \leqslant \exp\{ - \frac{(1-\varepsilon)}{\sqrt{2\pi}}\, T a_T^{-1} a_4(T) \exp( - \frac{a_4^2(T)}{2} )\}$$

$$\leqslant \exp\{ - \frac{(1-\varepsilon)}{\sqrt{\pi}} \sqrt{\pi(1+3\varepsilon)} \log\log T \} \leqslant (\log T)^{-1-\delta}$$

if $T$ is large enough, where $\delta$ is a suitable positive number. Put $T_k = \exp(k^{1+\rho}),\ k=1,2,\cdots;\rho>0$. Then

$$\sum_{k=1}^{\infty} P\{ Y_1(T_k) \leqslant a_4(T_k)\} < \infty ,$$

which implies this $P\{ Y_1(T_k) \leqslant a_4(T_k)\ \text{ i.o. }\} = 0$. Let $T_k \leqslant T < T_{k+1}$. Then

$$a_T^{1/2}\, Y_1(T) \geqslant \sup_{0 \leqslant t \leqslant T_k - a_{T_k}} (W(t+a_T) - W(t))$$

$$\geqslant \sup_{0 \leqslant t \leqslant T_k - a_{T_k}} ( W(t+a_{T_k}) - W(t)) - \sup_{0 \leqslant t \leqslant T_k - a_{T_k}} \sup_{0 \leqslant s \leqslant a_{T_{k+1}} - a_{T_k}} |W(t+s) - W(t)|.$$

Now (1.3.4) follows from (1.3.1) and Theorem 1.3.1 is proved.

**Corollary 1.3.1** *Let $a_T$ be as in Theorem 1.3.1 except that* (iii) *is replaced by the stronger condition*

$$\text{(iii}'\text{)} \qquad \lim_{T\to\infty} (\log T/a_T)^{1/2} / \log\log T = \infty .$$

*Then we have*

$$\lim_{T\to\infty} (Y_i(T) - (2\log T/a_T)^{1/2}) = 0 \qquad \text{a.s.} \quad i = 1, 2, 3, 4. \tag{1.3.8}$$

*Remark* 1.3.1 If Condition (iii $'$) does not hold true, then (1.3.8) does not hold as well. In fact, if

$$\lim_{T\to\infty} (\log T/a_T)^{1/2} / \log\log T = r > 0, \tag{1.3.9}$$

then

$$0 = \varliminf_{T\to\infty} (Y_i(T) - (2\log T/a_T)^{1/2}) < \varlimsup_{T\to\infty} (Y_i(T) - (2\log T/a_T)^{1/2})$$

$$= \frac{1}{r\sqrt{2}} \qquad \text{a.s.}$$

for $i = 1, 2, 3, 4$. If $r = 0$ in (1.3.9), but (iii) still holds true, then

$$\varlimsup_{T\to\infty} (Y_i(T) - (2\log T/a_T)^{1/2}) = \infty \qquad \text{a.s.} \quad i = 1, 2, 3, 4$$

but we have

$$\lim_{T\to\infty} ((2\log T/a_T)^{-1/2} Y_i(T) - 1) = 0 \qquad \text{a.s.} \quad i = 1, 2, 3, 4,$$

(see Theorem 1.1.1).

### 1.3.2 How fast is the rate of the inferior limit?

It is easy to see that from (1.2.6) we have

$$\varliminf_{T\to\infty} \sup_{a_T \le t \le T} |W(T) - W(T-t)|/d(T,t) = 0 \qquad \text{a.s.} \tag{1.3.10}$$

for $0 \le a_T \le T$. Hanson and Russo (1989) raised a question: What denominator should be used to obtain a positive, but finite, inferior limit? The partial answer to this problem has been given by Liu (1988).

**Theorem 1.3.2** (Liu 1988) *Let* $0<a_T\leqslant T$ *be a function of* $T$ *satisfying*

(iv) $\lim_{T\to\infty}(\log T/a_T)/\log\log T=r,\ 0\leqslant r\leqslant\infty$ .

*Then there exist constants* $C_1, C_2>0$ *such that*

$$C_1\sqrt{\frac{r}{1+r}}\leqslant I_i\leqslant C_2\sqrt{\frac{r}{1+r}}\qquad \text{a.s.}\qquad (1.3.11)$$

*where*

$$I_1=\varliminf_{T\to\infty}\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}|W(T)-W(T-t)|/d(T,t),$$

$$I_2=\varliminf_{T\to\infty}\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\ \sup_{0\leqslant s\leqslant t}|W(T)-W(T-s)|/d(T,t),$$

$$I_3=\varliminf_{T\to\infty}\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\beta(T,t)|W(T)-W(T-t)|,$$

$$I_4=\varliminf_{T\to\infty}\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\ \sup_{0\leqslant s\leqslant t}\beta(T,t)|W(T)-W(T-s)|,$$

*we can take* $C_1=\pi\sqrt{e-1}/(12\sqrt{e})$, $C_2=2e\sqrt{3.7/2.7}$ .

The proof of Theorem 1.3.2 will be given by a series of lemmas.

**Lemma 1.3.1**

$$I_i\leqslant 2e\qquad \text{a.s.}\qquad i=1,2,3,4.\qquad (1.3.12)$$

*Proof* It is enough to consider $I_2$ .

1° Note that $d(T,t)=\{2t(\log T/t+\log\log t)\}^{1/2}$ is an increasing function of $t\ (\leqslant T)$ when $T\geqslant e^e$. In order to prove (1.3.12), it need only to prove

$$\begin{aligned}&\varliminf_{T\to\infty}\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\ \sup_{0\leqslant s\leqslant t}|W(T)-W(T-s)|/d(T,t)\\ &\leqslant\varliminf_{T\to\infty}\sqrt{\log\log T}\sup_{0\leqslant t\leqslant T}|W(T)-W(T-t)|/d(T,t)\\ &\leqslant 2e\qquad \text{a.s.}\end{aligned}\qquad (1.3.13)$$

Let $T_n=\exp(n^2)$, and observe that

$$\sup_{0\leqslant t\leqslant T_n}|W(T_n)-W(T_n-t)|/d(T_n,t)$$

$$\leqslant \sup_{0<t\leqslant T_n-T_{n-1}} \frac{|W(T_n)-W(T_n-t)|}{d(T_n,t)}$$
$$+ \sup_{0\leqslant s\leqslant T_{n-1}} \frac{|W(T_{n-1})-W(T_{n-1}-s)|}{\{2(T_n-T_{n-1})\log\log(T_n-T_{n-1})\}^{1/2}}.$$

From Theorem 1.1.3 we have

$$\varlimsup_{n\to\infty} \sqrt{\log\log T_n} \sup_{0\leqslant s\leqslant T_{n-1}} \frac{|W(T_{n-1})-W(T_{n-1}-s)|}{\{2(T_n-T_{n-1})\log\log(T_n-T_{n-1})\}^{1/2}} = 0 \quad \text{a.s.} \tag{1.3.14}$$

If we can prove that

$$\varlimsup_{n\to\infty} \sqrt{\log\log T_n} \sup_{0\leqslant t\leqslant T_n-T_{n-1}} |W(T_n)-W(T_n-t)|/d(T_n,t) \leqslant 2e \quad \text{a.s.} \tag{1.3.15}$$

then, from (1.3.14) and (1.3.15), it follows that (1.3.13) holds true.

2° In order to prove (1.3.15), let $T=e^n$. We first prove that for any $\alpha\geqslant 2e$, there exists a positive integer $N=N(\alpha)$ such that for every $n\geqslant N$, we have

$$P\{\sqrt{\log\log T} \sup_{0<t\leqslant T} \frac{|W(T)-W(T-t)|}{d(T,t)} < \alpha\} \geqslant n^{-3e^2/((e-1)\alpha^2)}. \tag{1.3.16}$$

Denote $B(t)=W(T)-W(T-t)$. Then $\{B(t);\ 0\leqslant t\leqslant T\}$ is a Wiener process. By the property of independent increments we have

$$\begin{aligned}
I:&=P\{\sqrt{\log\log T} \sup_{0<t\leqslant T} |B(t)|/d(T,t)<\alpha\}\\
&\geqslant P\{\sqrt{\log n} \sup_{-\infty<k\leqslant n-1} \sup_{e^k\leqslant t\leqslant e^{k+1}} |B(t)|/\{2e^k(n-k+\log k)\}^{1/2}<\alpha\}\\
&\geqslant \prod_{k=-\infty}^{n-1} P\{\sqrt{\log n} \sup_{e^k\leqslant t\leqslant e^{k+1}} |B(t)-B(e^k)|\\
&\leqslant (\{2e^k(n-k+\log k)\}^{1/2}-\{2e^{k-1}(n-k+1+\log(k-1))\}^{1/2})\alpha\}.
\end{aligned} \tag{1.3.17}$$

Note that for large $n$ and every $k$, we have

$$\{2e^k(n-k+\log k)\}^{1/2}-\{2e^{k-1}(n-k+1+\log(k-1))\}^{1/2}$$
$$> \frac{\sqrt{e-1}}{2e}\{2e^k(e-1)(n-k+\log k)\}^{1/2}$$

and

$$\{ \sup_{e^k \leqslant t \leqslant e^{k+1}} | B(t) - B(e^k)| \} \overset{\mathcal{D}}{=} \{ \sup_{0 \leqslant s \leqslant e^k(e-1)} | B(s)| \},$$

so that

$$I \geqslant \prod_{k=-\infty}^{n-1} P\{ \sqrt{\log n} \sup_{0 \leqslant s \leqslant e^k(e-1)} |B(s)| < \frac{\sqrt{e-1}}{2e} (2e^k(e-1)$$

$$\cdot (n-k+\log k))^{1/2}\alpha \} = : \prod_{k=-\infty}^{n-1} J_{nk}. \tag{1.3.18}$$

By using inequality $2(1-\Phi(x)) \leqslant \exp(-x^2/2), x>0$, then

$$J_{nk} \geqslant 1 - 4(1 - \Phi( \frac{\sqrt{e-1}}{2e} \{ 2(n-k+\log k)/\log n\}^{1/2}\alpha ))$$

$$\geqslant 1 - 2\exp\{ -\frac{e-1}{4e^2} \cdot \frac{n-k+\log k}{\log n} \alpha^2 \}$$

$$\geqslant \exp\{ -2.6 \exp\{ - \frac{e-1}{4e^2} \frac{n-k+\log k}{\log n} \alpha^2 \}\}$$

for all $-\infty < k \leqslant n-1$. Thus we have

$$I \geqslant \exp\{ -2.6 \sum_{k=-\infty}^{n} \exp\{ - \frac{e-1}{4e^2} \frac{n-k+\log k}{\log n} \alpha^2\}\}. \tag{1.3.19}$$

If $n$ is large enough, then for any $\alpha \geqslant 2e$

$$2.6 \sum_{k=-\infty}^{n} \exp\{ - \frac{e-1}{4e^2} \frac{n-k+\log k}{\log n} \alpha^2 \} \leqslant \frac{1}{2} (1 - \exp\{ - \frac{(e-1)\alpha^2}{4e^2 \log n} \})^{-1}$$

uniformly in $n$. Therefore

$$I \geqslant \exp\{ - \frac{1}{2} (1 - \exp\{ - \frac{(e-1)\alpha^2}{4e^2 \log n} \})^{-1} \}. \tag{1.3.20}$$

But for any $\alpha \geqslant 2e$, there exists a number $N = N(\alpha)$ so that

$$1 - \exp( - \frac{(e-1)\alpha^2}{4e^2 \log n} ) \geqslant \frac{2}{3} \cdot \frac{(e-1)\alpha^2}{4e^2 \log n}$$

for every $n \geqslant N$. Substituting the last inequality into (1.3.20), we prove that (1.3.16) is true.

Finally, take $\alpha = 2e$, and from (1.3.16) we have

$$P\{ \sqrt{\log \log T_n} \sup_{0 < t \leqslant T_n - T_{n-1}} | W(T_n) - W(T_n - t)|/d(T_n, t) < 2e \}$$

$$\geqslant P\{\sqrt{\log\log T_n}\sup_{0<t\leqslant T_n}|W(T_n)-W(T_n-t)|/d(T_n,t)<2e\}$$
$$\geqslant n^{-3/(4(e-1))}.$$

By the Borel-Cantelli lemma, we prove that (1.3.15) holds true. The proof of Lemma 1.3.1 is completed.

**Lemma 1.3.2** *Suppose that* (iv) *is satisfied for* $0\leqslant r\leqslant 2.7$ . *Then we have*

$$I_i\leqslant \pi e\sqrt{r}\ /2\sqrt{e-1}\qquad \text{a.s.}\qquad i=1,2,3,4. \tag{1.3.21}$$

*Proof* It is enough to consider $I_2$ as well.

1° Let $0<\alpha\leqslant 2e$ , $T=e^n$. We first prove that for any $\varepsilon>0$ there exists a number $N=N(\varepsilon)$ such that for $n\geqslant N$ we have

$$P\{\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant t}|W(T)-W(T-s)|/d(T,t)<\alpha\}$$
$$\geqslant n^{-(r+\varepsilon)\pi^2e^2/4(e-1)\alpha^2}. \tag{1.3.22}$$

Denote $n_0=\min\{[\log a_T]\ ,n-1\}$, $B(t)=W(T)-W(T-t)$, $0\leqslant t\leqslant T$. Imitating the proof of (1.3.16), we have

$$P\{\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant t}|W(T)-W(T-s)|/d(T,t)<\alpha\}$$

$$\geqslant P\{\sqrt{\log n}\sup_{n_0\leqslant k\leqslant n-1}\sup_{e^k\leqslant t\leqslant e^{k+1}}\sup_{0\leqslant s\leqslant t}|B(s)|/(2e^k(n-k+\log k))^{1/2}<\alpha\}$$

$$\geqslant P\{\sqrt{\log n}\sup_{0\leqslant s\leqslant e^{n_0}}|B(s)|<\alpha(2e^{n_0-1}(n-n_0+1+\log(n_0-1)))^{1/2}\}$$

$$\cdot\prod_{k=n_0}^{n-1}P\{\sqrt{\log n}\sup_{e^k\leqslant s\leqslant e^{k+1}}|B(s)-B(e^k)|$$

$$<\alpha(\{2e^k(n-k+\log k)\}^{1/2}-\{2e^{k-1}(n-k+1+\log(k-1))\}^{1/2})\}$$

$$\geqslant P\{\sqrt{\log n}\sup_{0\leqslant s\leqslant e^{n_0}}|B(s)|<\alpha(2e^{n_0-1}(n-n_0+1+\log(n_0-1)))^{1/2}\}$$

$$\cdot\prod_{k=n_0}^{n-1}P\{\sqrt{\log n}\sup_{0\leqslant s\leqslant e^k(e-1)}|B(s)|<\alpha\frac{\sqrt{e-1}}{2e}\{2e^k(e-1)(n-k+\log k)\}^{1/2}\}$$

$$\geqslant P\{\sup_{0\leqslant s\leqslant 1}|B(s)|<\alpha\sqrt{2/e}\}(P\{\sup_{0\leqslant s\leqslant 1}|B(s)|<e^{-1}\alpha\sqrt{(e-1)/2}\})^{n-n_0}$$

$$\geqslant \exp\{-(n-n_0+1)\pi^2 e^2/(4(e-1)\alpha^2)\}, \tag{1.3.23}$$

in the last inequality we have used the well-known inequality (e. g. see Theorem 1.5.1 of Csörgő-Révész 1981)

$$P\{\sup_{0\leqslant s\leqslant 1} |B(s)|<u\}\geqslant \frac{4}{\pi}(e^{-\pi^2/8u^2}-\frac{1}{3}e^{-9\pi^2/8u^2})\geqslant e^{-\pi^2/8u^2}$$

for $u\leqslant 4.7$. From the hypothesis, it is clear that $(n-n_0+1)/\log n<r+\varepsilon$ when $n$ is large enough. So for any $\varepsilon>0$ there exists $N=N(\varepsilon)$ such that for $n\geqslant N$ we have

$$P\{\sqrt{\log\log T}\sup_{a_T\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant t}|W(T)-W(T-s)|/d(T,t)<\alpha\}$$

$$\geqslant \exp\{-(r+\varepsilon)\pi^2 e^2(\log n)/4(e-1)\alpha^2\}=n^{-(r+\varepsilon)\pi^2 e^2/4(e-1)\alpha^2},$$

i.e. (1.3.22) holds true.

$2^\circ$ For any $\varepsilon>0$, let $T_n=\exp(n^{1+\varepsilon})$. Imitating the proof of Lemma 1.3.1 and using (1.3.22), we have

$$I_2\leqslant \pi e\sqrt{(1+\varepsilon)(r+\varepsilon)}/2\sqrt{e-1} \qquad \text{a. s.}$$

Then (1.3.21) follows from the arbitrariness of $\varepsilon$ and the proof is completed.

**Lemma 1.3.3** *Suppose that* (iv) *is satisfied for* $0\leqslant r\leqslant\infty$. *Then we have*

$$I_i\geqslant \pi\sqrt{r(e-1)}/(12\sqrt{e}) \qquad \text{a. s.} \qquad i=1,2,3,4,$$

*for* $0\leqslant r\leqslant 1$, *and*

$$I_i\geqslant \pi\sqrt{e-1}\,(12\sqrt{e}) \qquad \text{a.s.} \qquad i=1,2,3,4,$$

*for* $1\leqslant r\leqslant\infty$.

*Proof* It is sufficient only to prove this for $I_3$.

$1^\circ$ Let $0<\varepsilon<1$, $T_n=\exp(n^{(1+r+\varepsilon)^{-1}})$. If $0\leqslant r\leqslant 1$, then for large $n$ we have

$$\inf_{T_{n-1}\leqslant T\leqslant T_n}\sup_{a_T\leqslant t\leqslant T}\beta(T,t)|W(T)-W(T-t)|$$

$$\geqslant \sup_{M_n\leqslant s\leqslant T_n}\beta(T_n,s)|W(T_n)-W(T_n-s)|$$

$$- \sup_{T_{n-1}\leqslant T\leqslant T_n} |W(T_n)-W(T)|/(2a_n'\log\log T_{n-1})^{1/2}$$

$$=: A_n - B_n, \tag{1.3.24}$$

where

$$M_n = \sup_{T_{n-1}\leqslant T\leqslant T_n} a_T + T_n - T_{n-1},\ a_n' = \inf_{T_{n-1}\leqslant T\leqslant T_n} a_T .$$

From (1.3.24), it is easy to conclude that

$$I_3 = \varliminf_{T\to\infty} \sqrt{\log\log T} \sup_{a_T\leqslant t\leqslant T} \beta(T,t)|W(T)-W(T-t)|$$

$$\geqslant \varliminf_{n\to\infty} \sqrt{\log\log T_{n-1}}\ A_n - \varlimsup_{n\to\infty} \sqrt{\log\log T_{n-1}}\ B_n . \tag{1.3.25}$$

Note that

$$\sqrt{\log\log T_{n-1}}\ B_n = (\{2(T_n - T_{n-1})(\log \frac{T_n}{T_n - T_{n-1}}$$
$$+\log\log(T_n - T_{n-1}))\}/2a_n')^{1/2}$$
$$\cdot \sup_{0\leqslant s\leqslant T_n - T_{n-1}} |W(T_n)-W(T_n - s)|/d(T_n, T_n - T_{n-1}),$$

and $T_n - T_{n-1} \to \infty$ as $n\to\infty$. Thus from Theorem 1.1.3 and

$$\lim_{n\to\infty} \{2(T_n - T_{n-1})(\log \frac{T_n}{T_n - T_{n-1}} + \log\log(T_n - T_{n-1}))\}/2a_n' = 0,$$

we have

$$\varlimsup_{n\to\infty} \sqrt{\log\log T_{n-1}}\ B_n = 0 \qquad \text{a.s.}$$

If we can prove that

$$\varliminf_{n\to\infty} \sqrt{\log\log T_{n-1}}\ A_n \tag{1.3.26}$$

$$\geqslant \frac{\pi\sqrt{(e-1)(1-\varepsilon)r}}{4\sqrt{e(1+(1+\varepsilon)r)}\ \sqrt{(1-\varepsilon)r/4+(1+\varepsilon)(1+r+\varepsilon)}} \qquad \text{a.s.}$$

then, from the arbitrariness of $\varepsilon$ and $0\leqslant r\leqslant 1$,

$$I_3 \geqslant \pi\sqrt{r(e-1)}\ /(12\sqrt{e}\ ) \qquad \text{a. s.} \tag{1.3.27}$$

If $1\leqslant r\leqslant\infty$, let $a_T' = \max\{a_T, T/\log T\}$. From (1.3.27), we have (note $\lim_{T\to\infty} (\log T/a_T')/\log\log T = 1$)

$$I_3 \geqslant \varliminf_{T\to\infty} \sqrt{\log\log T} \sup_{a_T \leqslant t \leqslant T} \beta(T,t)|W(T)-W(T-t)|$$
$$\geqslant \pi\sqrt{e-1}\ /12\sqrt{e}\ ).$$

2° In order to prove (1.3.26), let $T=e^{n'}$ (it is not necessary that $n'$ is integral). We first prove that for any $0<\varepsilon<1$ there exists $N=N(\varepsilon)$ such that for $n'\geqslant N$, any $\alpha>0$ and $0\leqslant r<\infty$, we have

$$P\{\sqrt{\log\log T} \sup_{a_T\leqslant t\leqslant T} \beta(T,t)|W(T)-W(T-t)|<\alpha\}$$
$$\leqslant n'^{(1/4-\pi^2(e-1)/16e(1+(1+\varepsilon)r)\alpha^2)(1-\varepsilon)r}. \tag{1.3.28}$$

It is clear that (1.3.28) holds true for $r=0$. Now we might assume that $r>0$. Let $\bar{n}=[n']$, $n_0\geqslant[\log a_T]$, $B(t)=W(T)-W(T-t)$, $0\leqslant t\leqslant T$. We have $\bar{n}\geqslant n_0+2$ when $n'$ is large enough. Imitating Lemma 1.3.1, we obtain

$$I:=P\{\sqrt{\log\log T} \sup_{a_T\leqslant t\leqslant T} \beta(T,t)|W(T)-W(T-t)|<\alpha\}$$
$$\leqslant P\{\sqrt{\log n} \sup_{n_0+2\leqslant k\leqslant n'} \sup_{e^{k-1}\leqslant t\leqslant e^k} |W(T) \tag{1.3.29}$$
$$-W(T-t)|/(2e^k(n'-k+\log n'))^{1/2}<\alpha\}.$$

Let

$$B_k=\{\sqrt{\log n'} \sup_{e^{k-1}\leqslant t\leqslant e^k} |W(T)-W(T-t)|/(2e^k(n'-k+\log n'))^{1/2}<\alpha\},$$

and

$$B'_{\bar{n}}=\{\sqrt{\log n'} \sup_{0\leqslant t\leqslant e^{\bar{n}}(1-1/e)} |B(t)|/(2e^{\bar{n}}(n'-\bar{n}+\log n'))^{1/2}<\alpha\}.$$

Let $P(\cdot\,\|\,x)$ stand for the distribution of Brownian motion starting at $x$. From the Markov property of a Wiener process, we have

$$P(\bigcap_{k=n_0+2}^{\bar{n}} B_k)=E(I(\bigcap_{k=n_0+2}^{\bar{n}-1} B_k)P(B'_{\bar{n}}\,\|\,B(e^{\bar{n}-1})))$$
$$\leqslant P(\bigcap_{k=n_0+2}^{\bar{n}-1} B_k)P(B'_{\bar{n}}).$$

Proceeding with successive substitution, we have

$$I\leqslant \prod_{k=n_0+2}^{\bar{n}} P\{\sqrt{\log n'} \sup_{0\leqslant t\leqslant e^k(1-1/e)} |B(t)|/(2e^k(n'-k+\log n'))^{1/2}<\alpha\}$$

$$=\prod_{k=n_0+2}^{\bar{n}} P\{\sup_{0\leqslant t\leqslant 1} |B(t)|<\alpha(2\frac{e}{e-1}(n'-k+\log n')/\log n')^{1/2}\} \tag{1.3.30}$$

$$\leqslant (P\{\sup_{0\leqslant t\leqslant 1}|B(t)|<\alpha(\frac{2e}{e-1}(n'-n_0-2+\log n')/\log n')^{1/2})^{\bar{n}-n_0-1}.$$

From the hypothesis of the lemma, it is easy to see that for any $\varepsilon>0$ there exists $N=N(\varepsilon)$such that for every $n'\geqslant N$

$$\bar{n}-n_0-1\geqslant(1-\varepsilon)\mathrm{r}\log n',\ (n'-n_0-2)/\log n'\leqslant(1+\varepsilon)r.$$

From Lemma 1.6.1 of Csörgő-Révész (1981), we have

$$I\leqslant(P\{\sup_{0\leqslant t\leqslant 1}|B(t)|<\alpha\sqrt{\frac{2e}{e-1}(1+(1+\varepsilon)r)}\})^{(1-\varepsilon)r\log n'}$$

$$\leqslant(\frac{4}{\pi}\exp(-\pi^2/8(\frac{2e}{e-1}\alpha^2(1+(1+\varepsilon)r))^{(1-\varepsilon)r\log n'}$$

$$\leqslant(\exp(\{\frac{1}{4}-\frac{\pi^2(e-1)}{16e(1+(1+\varepsilon)r)\alpha^2}\}(1-\varepsilon)r\log n')$$

$$=n'^{\{1/4-\pi^2(e-1)/16e(1+(1+\varepsilon)r)\alpha^2\}(1-\varepsilon)r}.$$

Thus, (1.3.28) holds true. Let $n=[n'^{1+r+\varepsilon}]$. Now by (1.3.28) we have

$$\sum_{n=1}^{\infty}P\left\{\sqrt{\log\log T_{n-1}}\,A_n\right.$$

$$\left.\leqslant\frac{\pi\sqrt{(e-1)(1-\varepsilon)r}}{4\sqrt{e(1+(1+\varepsilon)r)}\sqrt{(1-\varepsilon)r/4+(1+\varepsilon)(1+r+\varepsilon)}}\right\}$$

$$\leqslant\sum_{n=1}^{\infty}n^{-(1+\varepsilon)}<\infty.$$

From the Borel-Cantelli lemma, we obtain (1.3.26). The proof is completed.

*Proof of Theorem* 1.3.2. The theorem follows from Lemmas 1.3.1, 1.3.2 and 1.3.3.

## 1.4 How Large Are the Increments of a Two-Parameter Wiener Process?

As to the multi-parameter Wiener process, the discussion about the corresponding increments is usually much more complex; however, many

results on the increments of a Wiener process have been extended to the two-parameter case.

### 1.4.1 The Csörgő-Révész increments

Let $\{ W(s, t);\ 0 \leqslant s, t < \infty \}$ be a two-parameter Wiener process, i. e., for the rectangle $R = [x_1, x_2] \times [y_1, y_2] \subset R_+^2 := [0, \infty) \times [0, \infty)$, the $W$-measure

$$W(R) = W(x_2, y_2) - W(x_1, y_2) - W(x_2, y_1) + W(x_1, y_1)$$

satisfies the following conditions :

(i) $W(R) \in N(0, \lambda(R))$, where $\lambda(R) = (x_2 - x_1)(y_2 - y_1)$,

(ii) $W(0, y) \equiv W(x, 0) \equiv 0 \quad (0 \leqslant x, y < \infty)$,

(iii) $\{ W(s, t) \}$ is an independent increment process, that is $W(R_1)$, $W(R_2), \cdots, W(R_n)$ $(n = 2, 3, \cdots)$ are independent random variables, if $R_1$, $R_2, \cdots, R_n$ are disjoint rectangles,

(iv) the sample path $W(s, t; \omega)$ is continuous in $(s, t)$ with probability one.

Let $0 < a_T \leqslant T$, $b_T \geqslant T^{1/2}$ be non-decreasing functions of $T$ and denote

$$\delta_T = \{2 a_T (\log T/a_T + \log(\log b_T/a_T^{1/2} + 1) + \log\log T)\}^{-1/2}. \qquad (1.4.1)$$

Furthermore, let $L_T = L_T(a_T, b_T)$ (resp. $L_T^* = L_T^*(a_T, b_T)$) be the set of rectangles $R = [x_1, x_2] \times [y_1, y_2] \subset D_T(b_T)$ for which $\lambda(R) \leqslant a_T$ (resp. $\lambda(R) = a_T$), where

$$D_T(b_T) = \{ (x, y) : xy \leqslant T, 0 \leqslant x \leqslant b_T, 0 \leqslant y \leqslant b_T \}.$$

Csörgő and Révész (1978) discussed the increments of a two-parameter Wiener process and proved the following theorem :

**Theorem 1.4.1** (Csörgő, Révész 1978) *Suppose that*

(i) $T/a_T$ *is a non-decreasing function of* $T$,

(ii) $\delta_T$ *is a non-increasing function of* $T$,

(iii) *for any* $\varepsilon > 0$ *there exists a* $\theta_0 = \theta_0(\varepsilon) > 1$ *such that*

$$\overline{\lim_{k \to \infty}}\ \delta_{\theta^k}/\delta_{\theta^{k+1}} \leqslant 1 + \varepsilon \qquad (1.4.2)$$

*if* $1 < \theta \leqslant \theta_0$.

*Then*

$$\overline{\lim_{T \to \infty}} \sup_{R \in L_T} \delta_T |W(R)| = \overline{\lim_{T \to \infty}} \sup_{R \in L_T^*} \delta_T |W(R)| = 1 \quad \text{a.s.} \qquad (1.4.3)$$

*If we also have*

$$\text{(iv)} \quad \lim_{T \to \infty} \frac{\log T/a_T + \log(\log b_T / a_T^{1/2} + 1)}{\log \log T} = \infty ,$$

*then*

$$\lim_{T \to \infty} \sup_{R \in L_T} \delta_T |W(R)| = \lim_{T \to \infty} \sup_{R \in L_T^*} \delta_T |W(R)| = 1 \quad \text{a.s.} \qquad (1.4.4)$$

The following law of the iterated logarithm for $W(s, t)$ is a consequence of Theorem 1.4.1.

**Corollary 1.4.1** (Orey, Pruitt 1973; Park 1974, etc.) *We have*

$$\overline{\lim_{T \to \infty}} \sup_{0 \leqslant s, t \leqslant T, st = T} \frac{|W(s,t)|}{\sqrt{4T \log \log T}} = 1 \quad \text{a.s.} \qquad (1.4.5)$$

Lin (1985) first discussed the inferior limit problem for $\{W(R)\}$ and showed that if Condition (iv) fails, there exists a normalized factor for "$\underline{\lim}$" under an additional condition weaker than the above-mentioned one. That is analogous to a result of Csáki and Révész (1979), i.e. (1.1.10). Put

$$\lambda_T = \{2\, a_T (\log T / a_T + \log(\log b_T / a_T^{1/2} + 1))\}^{-1/2} .$$

**Theorem 1.4.2** (Lin 1985) *Suppose that*

(i) *$T/a_T$ is a non-decreasing function of $T$,*

(ii) *$\lambda_T$ is a non-increasing function of $T$,*

(iii) *for any $\varepsilon > 0$ there exists a $\theta_0 = \theta_0(\varepsilon) > 1$ such that*

$$\overline{\lim_{k \to \infty}} \lambda_{\theta^k} / \lambda_{\theta^{k+1}} \leqslant 1 + \varepsilon$$

*for* $1 < \theta \leqslant \theta_0$.

$$\text{(iv}') \qquad \lim_{T \to \infty} \frac{\log T / a_T + \log(\log b_T / a_T^{1/2} + 1)}{\log \log \log T} = \infty .$$

*Then*

$$\varliminf_{T\to\infty}\ \sup_{R\in LT}\lambda_T|W(R)|=\varliminf_{T\to\infty}\ \sup_{R\in LT^*}\lambda_T|W(R)|=1\qquad\text{a.s.}$$

*Proof* First, we prove that there exists a sequence of positive numbers $T_k\uparrow\infty$ $(k\to\infty)$ such that

$$\varliminf_{k\to\infty}\ \sup_{R\in L^*_{T_k}}\lambda_{T_k}|W(R)|\geqslant 1\qquad\text{a.s.}\tag{1.4.6}$$

Assume that $a_T/T\downarrow\rho<1$. Observe the region $D_T$ and insert the points $0=x_0<x_1\cdots<x_m\leqslant b_T(<x_{m+1})$ in the interval $[0,b_T]$ such that

$$(x_i-x_{i-1})y_i=a_T,\tag{1.4.7}$$

where $y_i=b_T\,(0\leqslant i\leqslant i_0)$ or $T/x_i\,(i_0\leqslant i\leqslant m+1)$ and

$$i_0:=\min\{i:T/x_i<b_T\}=[T/a_T]+1.$$

Obviously, $x'_{i0}=T^2/(b_T(T-a_T))$ is the solution of Eq. $(x'_{i_0}-T/b_T)T/x'_{i_0}=a_T$, and $x_{i_0}\leqslant x'_{i_0}$. Using (1.4.7), by induction we conclude that

$$x_i\leqslant x'_i:=T^{i-i_0+2}/(b_T(T-a_T)^{i-i_0+1})\qquad(i_0\leqslant i\leqslant m+1).$$

Since $x'_{m+1}>b_T$, i.e., $T^{m-i_0+3}/(b_T(T-a_T)^{m-i_0+2})>b_T$, we find

$$\begin{aligned}m&>i_0-2+(\log b_T^2/T)/\log(T/(T-a_T))\\&=[T/a_T]-1+(\log b_T^2/T)/\log(T/(T-a_T)).\end{aligned}\tag{1.4.8}$$

Put non-negative number sets $I_1=\{T:b_T/a_T^{1/2}<T/a_T\}$ and $I_2=\{T:b_T/a_T^{1/2}\geqslant T/a_T\}$. We may as well assume that both of them are unbounded. By Condition (iv$'$),

$$\lim_{T\to\infty,\,T\in I_1}T/a_T=\infty.$$

In fact, using (ii), we have $\lim_{T\to\infty}T/a_T=\infty$. And further, $\log b_T/a_T^{1/2}=o(T/a_T)$ as $T$ tends to infinity along $I_1$. Hence $\lambda_T/(2a_T\log T/a_T)^{-1/2}\to 1$. Noting $m>[T/a_T]-1$ and the tail behavior of a normal random variable, we have for any given $\varepsilon>0$

$$\begin{aligned}&P\{\sup_{R\in LT^*}\lambda_T|W(R)|\leqslant 1-\varepsilon\}\\&\leqslant\{1-\frac{\lambda_T}{(1-\varepsilon)\sqrt{2\pi a_T}}\exp(-\frac{1}{2}(1-\varepsilon)^2a_T\lambda_T^{-2})\}^m\end{aligned}\tag{1.4.9}$$

$$\leqslant \{ 1 - \frac{1}{4(1-\varepsilon)\sqrt{\log T/a_T}} \exp(-(1-2\varepsilon+2\varepsilon^2)\log T/a_T)\}^{[T/a_T]-1}$$

$$\leqslant \exp\{-\frac{[T/a_T]-1}{4(1-\varepsilon)\sqrt{\log T/a_T}}(T/a_T)^{-(1-2\varepsilon+2\varepsilon^2)}\}$$

$$\leqslant \exp\{-(T/a_T)^{\varepsilon}\}$$

for large $T \in I_1$. Because $(\log T/a_T)/\log\log\log T \to \infty$ as $T$ tends to infiniting along $I_1$, $\log T/a_T \geqslant (2/\varepsilon)\log\log\log T$, i.e.,

$$T/a_T \geqslant (\log\log T)^{2/\varepsilon} \tag{1.4.10}$$

for large $T \in I_1$. Set $T_0=0$ and $T_k=(1+k^{-1/2})^k$. It is easy to see that $T_k \uparrow \infty$ as $k \to \infty$. Since $\log(1+k^{-1/2}) \geqslant k^{-2/3}$ for large $k$, we have for $T_k \in I_1$

$$(T_k/a_{Tk})^{\varepsilon} \geqslant (\log\log T_k)^2 = \{\log(k\log(1+k^{-1/2}))\}^2 \tag{1.4.11}$$

$$\geqslant (\log k^{1/3})^2 \geqslant \log k^2$$

provided that $k$ is large enough. Inserting (1.4.11) into the exponent part of the right-hand side of (1.4.9) yields

$$\sum_{k=1}^{\infty}{}' P(\sup_{R \in L^*_{Tk}} \lambda_{Tk} | W(R)| \leqslant 1-\varepsilon) < \infty ,$$

where $\Sigma'$ denotes the sum over all $k$ satisfying $T_k \in I_1$. By the Broel-Cantelli lemma and the arbitrariness of $\varepsilon$, we obtain

$$\varliminf_{k \to \infty, T_k \in I_1} \sup_{R \in L^*_{Tk}} \lambda_{T_k} |W(R)| \geqslant 1 \qquad \text{a.s.} \tag{1.4.12}$$

If $T \in I_2$, then $b_T^2/T \geqslant b_T/a_T^{1/2}$. We discuss the cases of $\rho=0$ and $0<\rho<1$, respectively. If $\rho=0$, we have

$$(\log\frac{b_T^2}{T})/(\log\frac{T}{T-a_T}) \geqslant (\frac{T}{a_T}-1)\log\frac{b_T}{\sqrt{a_T}}$$

since $\log\frac{T}{T-a_T} = \log(1+\frac{a_T}{T-a_T}) \leqslant \frac{a_T}{T-a_T}$ for large $T$. Inserting this into (1.4.8), we obtain

$$P\{\sup_{R \in L^*_T} \lambda_T | W(R)| \leqslant 1-\varepsilon\}$$

$$\leqslant 1 - \frac{1}{\sqrt{2\pi}(1-\varepsilon)\sqrt{2(\log T/a_T + \log(\log b_T/\sqrt{a_T}+1))}}$$

$$\cdot \exp(-(1-\varepsilon)^2(\log T/a_T + \log(\log b_T/\sqrt{a_T}+1)))\}^m$$

$$\leqslant \exp\{-\frac{[T/a_T]-1+(T/a_T-1)\log b_T/\sqrt{a_T}}{2(1-\varepsilon)\sqrt{\pi(\log T/a_T+\log(\log b_T/\sqrt{a_t}+1))}}\cdot$$

$$\cdot((T/a_T)(\log b_T/\sqrt{a_T}+1))^{-(1-\varepsilon)^2}\}$$

$$\leqslant \exp\{-((T/a_T)(\log b_T/\sqrt{a_T}+1))^{\varepsilon}\}. \tag{1.4.13}$$

Similarly to (1.4.12), there holds

$$\varliminf_{k\to\infty,\, T_k\in I_2} \sup_{R\in L^*_{T_k}} \lambda_{T_k}|W(R)|\geqslant 1 \quad \text{a.s.} \tag{1.4.14}$$

If $0<\rho<1$, then $\log\dfrac{T}{T-a_T}\leqslant\log\dfrac{2}{1-\rho}$ for large $T$. Hence

$$(\log\frac{b_T^2}{T})/(\log\frac{T}{T-a_T})\geqslant(\log\frac{b_T}{\sqrt{a_T}})/(\log\frac{2}{1-\rho}).$$

Therefore, similarly to (1.4.13), there holds

$$P\{\sup_{R\in L_T^*}\lambda_T|W(R)|\leqslant 1-\varepsilon\}\leqslant\exp\{-(\log b_T/\sqrt{a_T})^{\varepsilon}\}. \tag{1.4.15}$$

From $0<\rho<1$ and Condition (iv′), we have $(\log\log b_T/\sqrt{a_T})/\log\log\log T \to \infty$, which implies that (1.4.14) is also true for $0<\rho<1$. These results being combined with (1.4.12), it follows that (1.4.6) holds for $\rho<1$.

In the following, we turn to the case of $\rho=1$. Now $a_T=T$ (cf. Csörgő and Révész 1981) and $(\log\log b_T/\sqrt{a_T})/\log\log\log T\to\infty$ by Condition (iv′). Let $a'_T=(1-1/\log\log T)T$,

$$\lambda'_T=(2a'_T\log(\log b_T/\sqrt{a'_T}+1))^{-1/2}$$

and suppose $L_T^{**}$ is $L_T^*$ in which $a_T(=T)$ is changed to $a'_T$. Clearly,

$$(\log\frac{b_T^2}{T})/(\log\frac{T}{T-a'_T})=(\log\log\log T)^{-1}\log\frac{b_T^2}{T}.$$

By a coniparison with (1.4.13), we find

$$P(\sup_{R\in L_T^{**}}\lambda'_T|W(R)|\leqslant 1-\varepsilon)$$

$$\leqslant \exp\left\{-\frac{-1+2(\log\log\log T)^{-1}\log b_T/\sqrt{T}}{4(1-\varepsilon)\sqrt{\log(\log b_T/\sqrt{a'_T}+1))}}(\log b_T/\sqrt{a'_T}+1)^{-(1-\varepsilon)^2}\right\}$$

$$\leqslant \exp\{-(\log b_T/\sqrt{T}\ )^{\varepsilon}\}$$

for large $T$. As (1.4.13) leads to (1.4.14), there holds

$$\varliminf_{k\to\infty}\sup_{R\in L^{**}_{Tk}}\lambda'_{Tk}|W(R)|\geqslant 1 \qquad \text{a.s.}$$

It is easy to see that $\lambda'_T/\lambda_T \to 1$ $(T\to\infty)$. Hence, further, we have

$$\varliminf_{k\to\infty}\sup_{R\in L^{**}_{Tk}}\lambda_{Tk}|W(R)|\geqslant 1 \qquad \text{a.s.} \tag{1.4.16}$$

Again let

$$a''_T=\frac{T}{\log\log T},\quad \lambda''_T=(2a''_T\log(\log b_T/\sqrt{a''_T}+1))^{-1/2},$$
$$\delta'_T=\{2a''_T(\log T/a''_T+\log(\log b_T/\sqrt{a''_T}+1)+\log\log T)\}^{-1/2},$$

and let $L'_T$ be $L_T$ in which $a_T$ is changed to $a''_T$. We are going to show that $T(\log\log T)^{-1}\log\log b_T/T^{1/2}$ is non-decreasing. Consider the function

$$f(T)=\frac{T}{\log\log T}\log\log\frac{C}{T^{1/2}}.$$

for $T$ satisfying $T^{1/2}\log T\leqslant C$ and $T>e^{e^e}$, where $C$ is a sufficiently large constant. It is easy to know that $f'(T)\geqslant 0$, i.e., $f(T)$ is a non-decreasing function. By Condition (iv$'$), we obtain $T^{1/2}\log T<b_T$ for large $T$. Then for sufficiently large $T_1<T_2$, and taking $C=b_{T_2}$ in $f(T)$, we have

$$\begin{aligned}\frac{T}{\log\log T_1}\log\log\frac{b_{T_1}}{\sqrt{T_1}} &\leqslant \frac{T_1}{\log\log T_1}\log\log\frac{b_{T_2}}{\sqrt{T_1}}\\ &\leqslant \frac{T_2}{\log\log T_2}\log\log\frac{b_{T_2}}{\sqrt{T_2}},\end{aligned}$$

and conclude that $\delta'_T$ is non-increasing. Furthermore, for any $\varepsilon>0$, under Assumption (iii) there exists a $\theta_0>1$ such that

$$\varlimsup_{k\to\infty}\delta'_{\theta^k}/\delta'_{\theta^{k+1}}\leqslant 1+\varepsilon$$

provided $1<\theta\leqslant\theta_0$. It is easy to see that $\lambda_T/\delta'_T\to 0$ $(T\to\infty)$. Then,

using Theorem 1.4.1, we have

$$\overline{\lim_{T \to \infty}} \sup_{R \in L'_T} \lambda_T | W(R)| = \overline{\lim_{T \to \infty}} \sup_{R \in L'_T} \delta'_T |W(R)| \frac{\lambda_T}{\delta'_T} = 0 \quad \text{a.s.} \quad (1.4.17)$$

Combining (1.4.17) with (1.4.16), we obtain for $\rho = 1$

$$\underline{\lim_{k \to \infty}} \sup_{R \in L^*_{Tk}} \lambda_{T_k} | W(R)|$$

$$\geqslant \underline{\lim_{k \to \infty}} \sup_{R \in L^{**}_{Tk}} \lambda_{T_k} | W(R)| - \overline{\lim_{k \to \infty}} \; 4 \sup_{R \in L'_{Tk}} \lambda_{T_k} | W(R)| \geqslant 1 \quad \text{a.s.}$$

Thus, we have proved that (1.4.6) is true for any $\rho \leqslant 1$.

Secondly, we prove that there is a subsequence $\{ k_k \}$ of positive integers such that

$$\overline{\lim_{k \to \infty}} \sup_{R \in L_{Tk_k}} \lambda_{T_{k_k}} | W(R)| \leqslant 1 \quad \text{a.s.} \quad (1.4.18)$$

For this purpose, we put

$$D'_T = \{ (x, y) : xy \leqslant T/a_T, 0 \leqslant x, y \leqslant b_T / a_T^{1/2} \},$$

$$L''_T = \{ R : R \subset D'_T, \lambda(R) \leqslant 1 \}$$

and

$$K(T) = 2(\log T/a_T + \log(\log b_T/\sqrt{a_T} + 1)).$$

From Condition (iv$'$) and $b_T \geqslant T^{1/2}$, we have $\lim_{k \to \infty} b_{T_k}/\sqrt{a_{T_k}} = \infty$. This implies that there is a subsequence $\{ k' \}$ of positive integers such that $b_{T_{k'}}/\sqrt{a_{T_{k'}}}$ tends to infinity monotonically. By Theorem 1.12.6 in Csörgő and Révész (1981), for any $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ such that

$$P(\sup_{R \in L''_{T_{k'}}} (K(T_{k'}))^{-1/2} | W(R)| \geqslant 1 + \varepsilon)$$

$$\leqslant c_\varepsilon (T_{k'}/a_{T_{k'}})(1 + \log T_{k'}/a_{T_{k'}})(1 + \log b_{T_{k'}}/\sqrt{a_{T_{k'}}})$$

$$\cdot \exp\left\{ - \frac{2(1+\varepsilon)^2}{2+\varepsilon} (\log T_{k'}/a_{T_{k'}} + \log(\log b_{T_{k'}}/\sqrt{a_{T_{k'}}} + 1)) \right\}$$

$$\leqslant 2c_\varepsilon ((T_{k'}/a_{T_{k'}}) \log b_{T_{k'}}/\sqrt{a_{T_{k'}}})^{-\varepsilon},$$

where $(T_{k'}/a_{T_{k'}}) \log b_{T_{k'}}/\sqrt{a_{T_{k'}}} \uparrow \infty$ $(k \to \infty)$. Define

$$k_n = \max\{k':(T_{k'}/a_{Tk'})\log b_{Tk'}/\sqrt{a_{Tk'}} \leqslant n^{2/\varepsilon}\}.$$

Then we have

$$\overline{\lim_{n\to\infty}} \sup_{R\in L''_{Tk_n}} (K(T_{k_n}))^{-1/2}|W(R)| \leqslant 1+\varepsilon \qquad \text{a.s.}$$

Since $L''_{Tk'}$ is non-decreasing, for $k_n < k' \leqslant k_{n+1}$, we have

$$\overline{\lim_{k'\to\infty}} \sup_{R\in L''_{Tk'}} (K(T_{k'}))^{-1/2}|W(R)|$$
$$\leqslant \overline{\lim_{n\to\infty}} \sup_{R\in L''_{k_{n+1}}} (K(T_{k_{n+1}}))^{-1/2}|W(R)| \left(\frac{K(T_{k_{n+1}})}{K(T_{k_n})}\right)^{1/2} \leqslant 1+\varepsilon \qquad \text{a.s.}$$

By the arbitrariness of $\varepsilon$, we get

$$\overline{\lim_{k'\to\infty}} \sup_{R\in L''_{Tk'}} (K(T_{k'}))^{-1/2}|W(R)| \leqslant 1 \qquad \text{a.s.}$$

Since $\sup_{R\in L''_{Tk'}} (K(T_{k'}))^{-1/2}|W(R)|$ has the same distribution as $\sup_{R\in L_{Tk'}} \lambda_{T_{k'}}|W(R)|$, there exists a subsequence $\{k_k\}$ of $\{k'\}$ such that (1.4.18) holds. From (1.4.6) and (1.4.18), we conclude that

$$\underline{\lim_{k\to\infty}} \sup_{R\in L_{Tk}} \lambda_{T_k}|W(R)| = \underline{\lim_{k\to\infty}} \sup_{R\in L^*_{Tk}} \lambda_{Tk}|W(R)| = 1 \qquad \text{a.s.} \qquad (1.4.19)$$

The remainder of the proof consists of filling in the gaps in the sequence $T_k$. We only investigate " $\sup_{R\in L^*_T}$ ". For any given $T>0$, we can find a $k$ such that $T_k < T \leqslant T_{k+1}$. Let $L_T(k) = \{R: R\subset D_T, \lambda(R) \leqslant a_T - a_{Tk}\}$. Write

$$\sup_{R\in L^*_T} \lambda_T|W(R)| \geqslant \sup_{R\in L^*_{Tk}} \lambda_T|W(R)| - 4\sup_{R\in L_T(k)} \lambda_T|W(R)|. \qquad (1.4.20)$$

Clearly, $\lambda_{Tk}/\lambda_{Tk+1} \to 1$ $(k\to\infty)$ by Condition (iii). From this and (1.4.19), we get

$$\underline{\lim_{k\to\infty}} \sup_{R\in L^*_{Tk}} \lambda_T|W(R)| = 1 \qquad \text{a.s.} \qquad (1.4.21)$$

Now we are ready to discuss $\sup_{R\in L_T^{(k)}} \lambda_T|W(R)|$. We may assume that $a_T - a_{T_k} > 0$. By Theorem 1.12.6 of Csörgő and Révész (1981), for $0<\varepsilon<1$

$$P\{\sup_{R\in L_T(k)} \lambda_T|W(R)| \geqslant \varepsilon\}$$

$$\leqslant \frac{c_\varepsilon T}{a_T - a_{Tk}} \left(1 + \log \frac{T}{a_T - a_{Tk}}\right)\left(1 + \log \frac{b_T}{\sqrt{a_T - a_{T_k}}}\right)$$

$$\cdot \exp\left\{-\frac{2\varepsilon^2}{2+\varepsilon} \cdot \frac{a_T}{a_T - a_{Tk}} \left(\log \frac{T}{a_T} + \log\left(\log \frac{b_T}{\sqrt{a_T}} + 1\right)\right)\right\}$$

$$\leqslant c_\varepsilon \exp\left\{-\frac{2\varepsilon^2}{2+\varepsilon} \cdot \frac{a_T}{a_T - a_{Tk}} \log \frac{T}{a_T} + \log\left(\frac{T}{a_T - a_{Tk}}\left(1 + \log \frac{T}{a_T - a_{Tk}}\right)\right)\right.$$

$$- \frac{2\varepsilon^2}{2+\varepsilon} \frac{a_T}{a_T - a_{Tk}} \log\left(\log \frac{b_T}{\sqrt{a_T}} + 1\right) + \log\left(1 + \log \frac{b_T}{\sqrt{a_T}}\right)$$

$$\left. + \frac{1}{2} \log \frac{a_T}{a_T - a_{Tk}} \right)\}$$

$$=: c_\varepsilon \exp(-d_1 + d_2 - d_3 + d_4). \tag{1.4.22}$$

Noting Condition (ii) and the definition of $T_k$, we have

$$1 \geqslant \frac{a_{Tk}}{a_{Tk+1}} \geqslant \frac{T_k}{T_{k+1}} = \left(1 + \frac{1}{\sqrt{k}}\right)^k \Big/ \left(1 + \frac{1}{\sqrt{k+1}}\right)^{k+1}$$

$$\geqslant \frac{\sqrt{k+1}}{\sqrt{k+1} + 1} \to 1, \ (k \to \infty), \tag{1.4.23}$$

and

$$\frac{a_T}{a_T - a_{Tk}} = \left(1 - \frac{a_{Tk}}{a_T}\right)^{-1} \geqslant \left(1 - \frac{a_{Tk}}{a_{Tk+1}}\right)^{-1} \geqslant \left(\frac{1}{\sqrt{k+1} + 1}\right)^{-1} > \sqrt{k}\,. \tag{1.4.24}$$

Therefore

$$d_4 = o(d_3), \ (T \to \infty). \tag{1.4.25}$$

Furthermore, $\log \frac{T}{a_T - a_{Tk}} = \log \frac{T}{a_T} + \log \frac{a_T}{a_T - a_{Tk}} = o(d_1)$ implies

$$d_2 = o(d_1), \quad (T \to \infty). \tag{1.4.26}$$

Substituting (1.4.25) and (1.4.26) into the right-hand side of (1.4.22), we find that it does not exceed.

$$c_\varepsilon \exp\left\{-\frac{1}{2}(d_1 + d_3)\right\} \tag{1.4.27}$$

$$\leqslant c_\varepsilon \exp\left\{-\frac{\varepsilon^2}{2+\varepsilon} \frac{a_T}{a_T - a_{Tk}} \left(\log \frac{T}{a_T} + \log\left(\log \frac{b_T}{\sqrt{a_T}} + 1\right)\right)\right\}$$

$$\leqslant c_\varepsilon \exp(-k),$$

provided that $T$ is large enough. It follows that

$$\sum_{k=1}^{\infty} P\Big( \sup_{R \in L_T(k)} \lambda_T | W(R)| \geq \varepsilon \Big) < \infty .$$

Thus

$$\overline{\lim_{k \to \infty}} \sup_{R \in L_T(k)} \lambda_T | W(R)| = 0 \quad \text{a.s.} \tag{1.4.28}$$

Putting (1.4.28) and (1.4.21) into (1.4.20), we obtain

$$\underline{\lim_{T \to \infty}} \sup_{R \in L_T}{}_{*} \lambda_T | W(R)| \geq 1 \quad \text{a.s.} \tag{1.4.29}$$

Certainly, we have further

$$\underline{\lim_{T \to \infty}} \sup_{R \in L_T} \lambda_T | W(R)| \geq 1 \quad \text{a.s.} \tag{1.4.30}$$

This completes the proof of the theorem by combining (1.4.29) and (1.4.30) with (1.4.19).

As a consequence of Theorem 1.4.2, we obtain a law of the iterated logarithm for $W(s, t)$ that is a companion to (1.4.5).

**Corollary 1.4.2** (Lacey 1989) *We have*

$$\underline{\lim_{T \to \infty}} \sup_{0 < s, t \leq T, st = T} \frac{| W(s, t)|}{\sqrt{2T \log \log T}} = 1 \quad \text{a.s.}$$

There is no analog of this limit for the one-parameter case. In fact, the law of the iterated logarithm of Chung (1948) states

$$\underline{\lim_{T \to \infty}} \left( \frac{8 \log \log T}{\pi^2 T} \right)^{1/2} \sup_{0 \leq t \leq T} |W(t)| = 1 \quad \text{a.s.}$$

**Corollary 1.4.3** (Kong 1987) *Suppose that $a_T$ and $\lambda_T$ satisfy conditions* (i), (ii), (iii) *in Theorem* 1.4.2 *and* (iv) *in Theorem* 1.4.1. *Then*

$$\lim_{T \to \infty} \sup_{R \in L_T} \lambda_T | W(R)| = \lim_{T \to \infty} \sup_{R \in L_T}{}_{*} \lambda_T |W(R)| = 1 \quad \text{a.s.} \tag{1.4.31}$$

*Proof* It is easy to see that $\delta_T$ satisfies all the conditions in Theorem 1.4.1 now, i.e., $\delta_T$ is non-increasing and

$$\overline{\lim_{k \to \infty}} \delta_{\theta^k}/\delta_{\theta^{k+1}} = \overline{\lim_{k \to \infty}} \lambda_{\theta^k}/\lambda_{\theta^{k+1}} \leqslant 1+\varepsilon$$

If $1<\theta\leqslant\theta_0$. Moreover, from Condition (iv)

$$\lim_{T \to \infty} \lambda_T/\delta_T = 1.$$

Consequently, (1.4.31) follows from Theorem 1.4.1 and 1.4.2.

*Remark* 1.4.1 If Condition (iv) in Theorem 1.4.1 and Condition (iv′) in Theorem 1.4.2 are replaced by Condition

$$(\text{iv}'') \lim_{T \to \infty} \frac{\log T/a_T + \log(\log b_T/a_T^{1/2}+1)}{\log \log T} = r \qquad 0\leqslant r<\infty ,$$

Kong (1987) have also shown that : Under Conditions (i) and (ii) in Theorem 1.4.2 and (iv″), we have

$$\underline{\lim_{T \to \infty}} \sup_{R \in L_T^*} \delta_T |W(R)| = \underline{\lim_{T \to \infty}} \sup_{R \in L_T} \delta_T |W(R)| = \sqrt{r/(r+1)} \qquad \text{a.s.}$$

$$\overline{\lim_{T \to \infty}} \sup_{R \in L_T^*} \delta_T |W(R)| = \overline{\lim_{T \to \infty}} \sup_{R \in L_T} \delta_T |W(R)| = 1 \qquad \text{a.s.}$$

$$\overline{\lim_{T \to \infty}} \sup_{R \in L_T^*} \lambda_T |W(R)| = \overline{\lim_{T \to \infty}} \sup_{R \in L_T} \lambda_T |W(R)| = \sqrt{(r+1)/r} \qquad \text{a.s.}$$

### 1.4.2 Lag increments

The lag increments of a two-parameter Wiener process were discussed by Lu (1991 a, b). A theorem corresponding to Theorem 1.1.3 is as follows.

**Theorem 1.4.3** (Lu 1991a) *Let $b_T \geqslant T^{1/2}$ be a non-decreasing function of $T$. Denote*

$$L_T^*(t) = \{ R : R \subset D_T(b_T),\ \lambda(R) = t \},$$

$$L_T(t) = \{ R : R \subset D_T(b_T),\ \lambda(R) \leqslant t \},$$

$$d^*(T, t) = \{ 2t(\log T/t + \log(\log b_T/t^{1/2}+1) + \log \log t ) \}^{1/2}.$$

*Suppose that* $\gamma_T = d^*(T, T)^{-1}$ *satisfies*

(ii ′) $\gamma_T$ *is a non-increasing function of* $T$,

(iii ′) *for any* $\varepsilon > 0$ *there exists a* $\theta_0 = \theta_0(\varepsilon) > 1$ *such that*

$$\overline{\lim_{k \to \infty}} \; \gamma_{\theta^k} / \gamma_{\theta^{k+1}} \leqslant 1 + \varepsilon$$

*for* $1 < \theta \leqslant \theta_0$. *Then*

$$\lim_{T \to \infty} \sup_{0<t<T} \sup_{R \in L_T^*(t)} |W(R)|/d^*(T, t) = 1 \quad \text{a.s.} \tag{1.4.32}$$

$$\lim_{T \to \infty} \sup_{0<t \leqslant T} \sup_{R \in L_T(t)} |W(R)|/d^*(T, t) = 1 \quad \text{a.s.} \tag{1.4.33}$$

*Proof* 1° In order to prove

$$\overline{\lim_{T \to \infty}} \sup_{0<t \leqslant T} \sup_{R \in L_T(t)} |W(R)|/d^*(T, t) \leqslant 1 \quad \text{a.s.} \tag{1.4.34}$$

we take a real number $\theta > 1$ such that $1 < 2(1+\varepsilon)^2/((2+\varepsilon)\theta) = : 1 + 2\varepsilon'$. Let $1 < v \leqslant \theta_0$. Define $T_n = v^n$, $k_n = [(n+1)\log v/\log\theta] + 1$ and let $t_k$, $k_\theta$ be the same as in Theorem 1.1.3. Take $\beta = 2/\varepsilon'$ and $k_n' = [(\log\theta)^{-1}\log(T_{n+1}(\log T_n)^{-\beta})]$.

For any $T > 0$, there exists a $T_n$ such that $T_n < T \leqslant T_{n+1}$. We have

$$\begin{aligned} &\sup_{0<t \leqslant T} \sup_{R \in L_T(t)} |W(R)|/d^*(T, t) \\ &\leqslant \sup_{-\infty<k \leqslant k_n - 1} \sup_{R \in L_{T_{n+1}}(t_k, t_{k+1})} \{2t_k(\log T_n/t_{k+1} + \log(\log b_{T_n}/t_{k+1}^{1/2} + 1) \\ &\qquad + \log\log t_k)\}^{-1/2}|W(R)| \\ &= : \sup_{-\infty<k \leqslant k_n - 1} A_{nk}, \end{aligned} \tag{1.4.35}$$

where $L_{T_{n+1}}(t_k, t_{k+1}) = \{R : R \subset D_{T_{n+1}}(b_{T_{n+1}}),\ t_k \leqslant \lambda(R) = t \leqslant t_{k+1}, t \leqslant T_{n+1}\}$. Noting

$$L_{T_{n+1}}(t_k, t_{k+1}) \subset L_{T_{n+1}}(t_{k+1}) \tag{1.4.36}$$

and using Theorem 1.12.6 of Csörgő-Révész (1981) and Condition (iii ′), we have

$$P\{A_{nk} \geqslant 1 + \varepsilon\} \tag{1.4.37}$$

$$\leqslant P\{\sup_{R\in LT_{n+1}(t_{k+1})}|W(R)|/t_{k+1}^{1/2}\geqslant(1+\varepsilon)(2\theta^{-1}(\log T_n/t_{k+1}$$

$$+\log(\log b_{T_n}/t_{k+1}^{1/2}+1)+\log\log t_k))^{1/2}\}$$

$$\leqslant c\frac{T_{n+1}}{t_{k+1}}(1+\log\frac{T_{n+1}}{t_{k+1}})(1+\log\frac{b_{T_{n+1}}}{\sqrt{t_{k+1}}})\exp\{-\frac{2(1+\varepsilon)^2}{(2+\varepsilon)\theta}$$

$$\cdot(\log\frac{T_n}{t_{k+1}}+\log(\log\frac{b_{T_n}}{\sqrt{t_{k+1}}}+1)+\log\log t_k)\}$$

$$\leqslant c(\frac{T_{n+1}}{t_{k+1}})^{-\varepsilon'}(1+\log\frac{T_{n+1}}{t_{k+1}})(\log T_n)^{\varepsilon}(\log t_k)^{-1-2\varepsilon'}$$

for large $n$. Imitating the proof of Theorem 1.1.3, we have

$$\sum_{n=1}^{\infty}P\{\sup_{-\infty<k\leqslant k_n-1}A_{nk}\geqslant 1+\varepsilon\}<\infty\ ,$$

and (1.4.34) follows from the Borel-Cantelli lemma and (1.4.35).

2° In order to prove (1.4.32) and (1.4.33), it is sufficient to show that

$$I:=\varliminf_{T\to\infty}\sup_{0<t\leqslant T}\sup_{R\in L_T^*(t)}|W(R)|/d^*(T,t)\geqslant 1\qquad\text{a.s.}\quad(1.4.38)$$

Denote $n=[T]$. Let

$$A_{i+1}=\left[(\frac{n-1}{n})^{i+1}b_n,(\frac{n-1}{n})^{i}b_n\right]\times\left[0,\frac{n^{i+1}}{(n-1)^ib_n}\right]\qquad i=0,1,\cdots,l,$$

where $l=\max\{i:n^{i+1}<(n-1)^ib_n^2\}$. Clearly, $A_i\in L_n^*(1)$, $1\leqslant i\leqslant l+1$, $l\sim cn\log b_n^2/n$ and we have

$$I\geqslant\varliminf_{T\to\infty}\sup_{R\in L_T^*(1)}|W(R)|/d^*(T,1)$$

$$\geqslant\varliminf_{T\to\infty}\sup_{R\in L_n^*(1)}\{2(\log(n+1)+\log(\log b_{n+1}+1))\}^{-1/2}|W(R)|$$

$$\geqslant\varliminf_{n\to\infty}\max_{1\leqslant i\leqslant l+1}\{2(\log(n+1)+\log(\log b_{n+1}+1))\}^{-1/2}|W(A_i)|.$$

By using the well-known estimate of the tail probability of the normal distribution, it follows that

$$\sum_{n=1}^{\infty}P\{\max_{1\leqslant i\leqslant l+1}|W(A_i)|\leqslant(2(1-\varepsilon)(\log(n+1)+\log(\log b_{n+1}+1)))^{1/2}\}$$

$$\leqslant \sum_{n=1}^{\infty}\{1-\exp(-(1-\varepsilon)(\log(n+1)+\log(\log b_{n+1}+1)))\}^{cn\log(b_n^2/n)}$$

$$\leqslant \sum_{n=1}^{\infty} \exp\{-c(n\log b_n)^{\varepsilon/2}\}<\infty .$$

Hence, by using the Borel-Cantelli lemma, the above inequality implies (1.4.38). The proof of Theorem 1.4.3 is completed.

### 1.4.3 General form of the increments

One hopes to get a general result similar to Theorem 1.1.4. The following theorem not only implies Theorem 1.4.1, etc., but also weakens the conditions to a great extent.

**Theorem 1.4.4** (Lin, Lu 1990) *Let $a_T$, $c_T$ and $d_T$ be non-negative functions of $T$ with $a_T+d_T \geqslant c_T \to \infty$ as $T\to\infty$ and let $b_T(\geqslant T^{1/2})$ be a non-decreasing function of $T$ which satisfies Conditions (ii$'$) and (iii$'$) of Theorem 1.4.3. If there exists a constant $A>0$ such that for any $T>2$*

$$a_T+d_T \leqslant A(a_{T-1}+d_{T-1}), \tag{1.4.39}$$

*then*

$$\overline{\lim_{T\to\infty}} \sup_{0\leqslant t}\sup_{0<s}\sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T, s)=1 \quad \text{a.s.} \tag{1.4.40}$$

$$\overline{\lim_{T\to\infty}} \sup_{R\in L^*_{a_T+d_T}(a_T)} \beta^*(a_T+d_T, a_T)|W(R)|=1 \quad \text{a.s.} \tag{1.4.41}$$

*where*

$$\beta^*(M,m)=\{2m(\log M/m+\log(\log b_M/m^{1/2}+1)+\log\log M)\}^{-1/2}.$$

*Furthermore, if*

$$\sum_{n=1}^{\infty}\exp\{-(\frac{a_n+d_n}{a_n}(\log\frac{\widetilde{b}_n}{\sqrt{a_n}}+1))^{\varepsilon}/(\log(a_n+d_n))^{1-\varepsilon}\}<\infty \tag{1.4.42}$$

*for any $0<\varepsilon<1$, where $\widetilde{b}_n=b_{a_n+d_n}$, and*

$$\lim_{T\to\infty} a_T/a_{[T]}=\lim_{T\to\infty} d_T/d_{[T]}=1, \tag{1.4.43}$$

*then*

$$\lim_{T\to\infty} \sup_{0\leq t} \sup_{0<s} \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T, s)=1 \quad \text{a.s.} \qquad (1.4.44)$$

$$\lim_{T\to\infty} \sup_{0\leq t\leq d_T} \sup_{R\in L^*_{t+a_T}(a_T)} \beta^*(a_T+d_T, a_T)|W(R)|=1 \quad \text{a.s.} \qquad (1.4.45)$$

*Remark* 1.4.3 Taking $d_T=T-a_T$ in (1.4.41), we have

$$\overline{\lim_{T\to\infty}} \sup_{R\in L^*_T} \delta_T|W(R)|=\overline{\lim_{T\to\infty}} \sup_{R\in L^*_T(a_T)} \beta^*(T, a_T)|W(R)|=1 \quad \text{a.s.} \quad (1.4.46)$$

without any condition on $a_T$. Hence (1.4.3) is true under Conditions (ii $'$) and (iii$'$), i.e., for Theorem 1.4.1 the condition that $a_T$ is non-decreasing is unnecessary, Condition (i) can be removed and the Conditions (ii) and (iii) can be replaced by weaker Conditions (ii $'$) and (iii $'$). If, in addition, we have

$$\sum_{n=1}^{\infty} \exp\{-((n/a_n)(\log b_n/a_n^{1/2}+1))^{\varepsilon}/(\log n)^{1-\varepsilon}\}<\infty \qquad (1.4.42\,')$$

for any $0<\varepsilon<1$ and

$$\lim_{T\to\infty} a_T/a_{[T]}=1, \qquad (1.4.43\,')$$

then (1.4.4) is true.

Obviously (1.4.40) implies that

$$\overline{\lim_{T\to\infty}} \sup_{0\leq t\leq T-a_T} \sup_{R\in L_{t+a_T}(a_T)} |W(R)|/d^*(t+a_T, a_T)\leq 1 \quad \text{a.s.} \qquad (1.4.47)$$

$$\overline{\lim_{T\to\infty}} \sup_{0\leq t\leq T-a_T} \sup_{R\in L^*_{t+a_T}(a_T)} |W(R)|/d^*(t+a_T, a_T)\leq 1 \quad \text{a.s.} \qquad (1.4.48)$$

If conditions (ii $'$), (iii $'$) and condition $a_T\to\infty$ are satisfied, we have

$$\overline{\lim_{T\to\infty}} \sup_{0\leq t\leq T-a_T} \sup_{R\in L_{t+a_T}(a_T)} |W(R)|/d^*(t+a_T, a_T)=1 \quad \text{a.s.} \qquad (1.4.49)$$

$$\overline{\lim_{T\to\infty}} \sup_{0\leq t\leq T-a_T} \sup_{R\in L^*_{t+a_T}(a_T)} |W(R)|/d^*(t+a_T, a_T)=1 \quad \text{a.s.} \qquad (1.4.50)$$

Furthermore, if Conditions (1.4.42 $'$) and (1.4.43 $'$) are added, then $\overline{\lim}$ can be replaced by lim by (1.4.49) and (1.4.50). These conclusions improve a theorem of Lu (1991 a) which is an analogy to Theorem 3.2B of Hanson and Russo (1983) in two-parameter case.

We will need the following lemma in the proof of Theorem 1.4.4.

**Lemma 1.4.1** *Suppose that $\gamma_T$ satisfies the Conditions* (ii $'$) *and* (iii $'$) *of Theorem* 1.4.3. *Then we have*

$$\lim_{a\to\infty} \sup_{R\in Lv(v-u),\ a\leqslant v-u} |W(R)|/d^*(v, v-u) = 1 \quad \text{a.s.} \qquad (1.4.51)$$

The proof is quite similar to that of Corollary 1. 1. 1 and will not be presented here.

*Proof of Theorem* 1.4.4.

1° We prove

$$\overline{\lim_{T\to\infty}} \sup_{0\leqslant t} \sup_{0<s} \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T, s) \leqslant 1 \quad \text{a.s.} \qquad (1.4.52)$$

It is clear that we can assume that $c_T \to \infty$ non-decreasingly as $T\to\infty$, otherwise, we consider $c_T^* = \inf_{T\leqslant t} c_T$ instead of $c_T$.

For given $B>0$, we write

$$\sup_{0\leqslant t} \sup_{0<s} \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T, s) \qquad (1.4.53)$$

$$= \Big(\sup_{0\leqslant t} \sup_{B<s} \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T, s)\Big)$$

$$\vee \Big(\sup_{0\leqslant t\leqslant 1} \sup_{0<s\leqslant B} \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T, s)\Big)$$

$$\vee \Big(\sup_{1\leqslant t} \sup_{0<s\leqslant B} \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T, s)\Big)$$

$$=: I_1 \vee I_2 \vee I_3 .$$

Using Lemma 1.4.1, for any given $\varepsilon>0$ there exists a large $B=B(\varepsilon)$, such that $I_1\leqslant 1+\varepsilon$ a. s. Let $\theta>1$ be close to 1 . Using Theorem 1. 12. 6 of Csörgő and Révész (1981) and Condition (iii $'$), for large $T$, we have

$$P\{ I_3 \geqslant 1+\varepsilon \}$$

$$\leqslant \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P\Big\{ \sup_{\theta^i\leqslant t<\theta^{i+1}} \sup_{B\theta^{-(j+1)}<s\leqslant B\theta^{-j}} \sup_{R\in L_{t+s}(s)} |W(R)|/d^*(t+s\vee c_T, s) \geqslant 1+\varepsilon \Big\}$$

$$\leqslant \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P\Big\{ \sup_{R\in L_{\theta^{i+1}+B\theta^{-j}}(B\theta^{-j})} |W(R)| \geqslant (1+\varepsilon)(2B\theta^{-(j+1)}(\log\frac{\theta^i + c_T}{B\theta^{-j}}$$

$$+ \log(\log b_{\theta^i+B\theta^{-(j+1)}}/\sqrt{B\theta^{-j}} + 1) + \log\log B\theta^{-(j+1)}))^{1/2} \Big\}$$

$$\leqslant c\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{i+1}+B\theta^{-j}}{B\theta^{-j}}\left(1+\log\frac{\theta^{i+1}+B\theta^{-j}}{B\theta^{-j}}\right)(1+\log b_{\theta^{i+1}+B\theta^{-j}}/\sqrt{B\theta^{-j}})$$

$$\cdot\left\{\frac{\theta^{i}+c_T}{B\theta^{-j}}(1+\log b_{\theta^{i}+B\theta^{-(j+1)}}/\sqrt{B\theta^{-j}})\log B\theta^{-(j+1)}\right\}^{-(1+\varepsilon)}$$

$$\leqslant c\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(\theta^{i}+c_T)^{-\varepsilon/2}\theta^{-j\varepsilon/2}\leqslant c\,c_T^{-\varepsilon/2}\to 0 \quad \text{as } T\to\infty .$$

Similarly, we have $P\{I_2\geqslant 1+\varepsilon\}\to 0$ as $T\to\infty$. Since $I_2$ and $I_3$ both are non-increasing functions of $T$, we obtain

$$\overline{\lim_{T\to\infty}}(I_2\vee I_3)\leqslant 1+\varepsilon \qquad \text{a.s.}$$

Thus (1.4.52) is proved.

2° Now, in order to prove (1.4.40) and (1.4.41), we need only to to show that

$$\overline{\lim_{T\to\infty}}\sup_{R\in L^{*}_{a_T+d_T}(a_T)}\beta^{*}(a_T+d_T,a_T)|W(R)|\geqslant 1 \qquad \text{a.s.} \quad (1.4.54)$$

Define $N_1=1$ and

$$N_k=\max\{n: a_n+d_n\leqslant\theta^k\} \qquad k\geqslant 2,$$

By Condition (1.4.39), we have

$$\theta^k/A<(a_{N_k+1}+d_{N_k+1})/A\leqslant a_{N_k}+d_{N_k}\leqslant\theta^k . \qquad (1.4.55)$$

First of all, we assume that

$$\rho_1:=\overline{\lim_{N\to\infty}}\,a_N/(a_N+d_N)<1.$$

Define $L=L(k)$ to be the largest integer for which we have

$$(a_{N_k}+d_{N_k})^{L+1}/d^{L}_{N_k}\widetilde{b}_{N_k}\leqslant\widetilde{b}_{N_k}$$

for any given $k$, and the rectangles

$$S_i(k)=[x_1(i),\ x_2(i)]\times[y_1(i),\ y_2(i)]$$

$$=\left[\left(\frac{d_{N_k}}{a_{N_k}+d_{N_k}}\right)^{i+1}\widetilde{b}_{N_k},\left(\frac{d_{N_k}}{a_{N_k}+d_{N_k}}\right)^{i}\widetilde{b}_{N_k}\right]$$

$$\times\left[\frac{(a_{N_{k-1}}+d_{N_{k-1}})(a_{N_k}+d_{N_k})^{i+1}}{d_{N_k}^{i+1}\,\tilde{b}_{N_k}},\ \frac{(a_{N_k}+d_{N_k})^{i+1}}{d_{N_k}^{i}\,\tilde{b}_{N_k}}\right],$$

where $i=0,1,\cdots,L$. We observe that

$$a_{N_{k-1}}+d_{N_{k-1}}=x_1(i)y_1(i)<x_2(i)y_2(i)=a_{N_k}+d_{N_k},$$

$$0<x_1(i)<x_2(i)\leqslant\tilde{b}_{N_k},\ 0<y_1(i)<y_2(i)\leqslant\tilde{b}_{N_k},\qquad i=0,1,\cdots,L.$$

Hence, putting $\overline{D}_{N_k}=D_{a_{N_k}+d_{N_k}}$, we have

$$S_i(k)\subset\overline{D}_{N_k}-\overline{D}_{N_{k-1}},$$

and for each $k$, $S_i(k)$, $i=0,1,\cdots,L$, are disjoint rectangles. By (1.4.55) and the assumption $\rho_1<1$, there exists $k_0>1$ such that

$$(a_{N_{k-1}}+d_{N_{k-1}})/d_{N_k}\leqslant\theta^{k-1}/(\theta^k/A)(1-\rho_1-\delta)$$

for some $0<\delta<1-\rho_1$ and $k\geqslant k_0$. Therefore for any given $0<\varepsilon<1$,

$$(1-\varepsilon)a_{N_k}\leqslant\lambda(S_i(k))\leqslant a_{N_k}\tag{1.4.56}$$

for $k\geqslant k_0$ and every $i$, provided that $\theta$ is large enough. Whence we have for $k\geqslant k_0$

$$\begin{aligned}
&P\{\max_{0\leqslant i\leqslant L}\beta^*(a_{N_k}+d_{N_k},a_{N_k})|W(S_i(k))|\geqslant 1-\varepsilon\}\\
&\geqslant 1-(1-2\{1-\Phi(\sqrt{1-\varepsilon}\,(2(\log\frac{a_{N_k}+d_{N_k}}{a_{N_k}}+\log(\log\frac{\tilde{b}_{N_k}}{\sqrt{a_{N_k}}}+1)\\
&\quad+\log\log(a_{N_k}+d_{N_k})))^{1/2})\})^{L+1}\qquad(1.4.57)\\
&\geqslant 1-(1-\{\frac{a_{N_k}+d_{N_k}}{a_{N_k}}(\log\frac{\tilde{b}_{N_k}}{\sqrt{a_{N_k}}}+1)\log(a_{N_k}+d_{N_k})\}^{-(1-\varepsilon)})^{L+1}\\
&\geqslant 1-\exp\{-\frac{a_{N_k}}{a_{N_k}+d_{N_k}}\cdot\frac{1}{\log\tilde{b}_{N_k}/\sqrt{a_{N_k}}+1}\cdot\frac{1}{\log(a_{N_k}+d_{N_k})})^{-\varepsilon}(L+1)\}.
\end{aligned}$$

Since $L+1\geqslant c\dfrac{a_{N_k}+d_{N_k}}{a_{N_k}}\log\dfrac{\tilde{b}_{N_k}^2}{a_{N_k}+d_{N_k}}$ by the definition of $L$, we obtain

$$\sum_{k=2}^{\infty} P\{\max_{0\leqslant i\leqslant L} \beta^*(a_{N_k}+d_{N_k}, a_{N_k})|W(S_i(k))| \geqslant 1-\varepsilon\}$$

$$\geqslant c\sum_{k=2}^{\infty}\left(\frac{a_{N_k}+d_{N_k}}{a_{N_k}}\right)^{\varepsilon}\frac{1}{\log(a_{N_k}+d_{N_k})}\left\{\log\frac{\widetilde{b}^2_{N_k}}{a_{N_k}+d_{N_k}}\Big/\left(\log\frac{\widetilde{b}^2_{N_k}}{a_{N_k}+d_{N_k}}\right.\right.$$

$$\left.\left.+\log\frac{a_{N_k}+d_{N_k}}{a_{N_k}}\right)\right\}$$

$$\geqslant c\sum_{k=2}^{\infty}\frac{1}{\log(a_{N_k}+d_{N_k})}=\infty\ , \tag{1.4.58}$$

which implies

$$\overline{\lim_{k\to\infty}}\max_{0\leqslant i\leqslant L}\beta^*(a_{N_k}+d_{N_k}, a_{N_k})|W(S_i(k))|\geqslant 1-\varepsilon \qquad \text{a.s.}$$

by independence of $\{W(S_i(k))\}$ for each $k$ and the Borel-Cantelli lemma. Denote $a'_{N_k}=\lambda(S_i(k))$. $\overline{a}_{N_k}:=a_{N_k}-a'_{N_k}\leqslant \varepsilon a_{N_k}$ by (1.4.56). Then

$$\overline{\lim_{T\to\infty}}\sup_{R\in L^*_{a_T+d_T}(a_T)}\beta^*(a_T+d_T, a_T)|W(R)|$$

$$\geqslant \overline{\lim_{k\to\infty}}\{\max_{0\leqslant i\leqslant L}\beta^*(a_{N_k}+d_{N_k}, a_{N_k})|W(S_i(k))|$$

$$-2\sup_{R\in L_{a_{N_k}+d_{N_k}}(\overline{a}_{N_k})}\beta^*(a_{N_k}+d_{N_k}, a_{N_k})|W(R)|\}$$

$$\geqslant 1-3\varepsilon \qquad \text{a.s.}$$

where a conclusion similar to (1.4.52) was applied. Consequently, we get (1.4.54) when $\rho_1<1$.

Next, considering the case of $\rho_2:=\lim_{N\to\infty}\dfrac{a_N}{a_N+d_N}=1$. Define $L=L(k)$ also to be the largest integer for which we have

$$(a_{N_k}+d_{N_k})^{1/2}M^{L+1}\leqslant\widetilde{b}_{N_k}$$

with $M(=2/\varepsilon)>1$, and define the rectangles

$$S_i(k)=[x_1(i),\ x_2(i)]\times[y_1(i),\ y_2(i)]$$

$$=[(a_{N_k}+d_{N_k})^{1/2}M^i, (a_{N_k}+d_{N_k})^{1/2}M^{i+1}]\times[(a_{N_{k-1}}+d_{N_{k-1}})$$

$$(a_{N_k}+d_{N_k})^{-1/2}M^{-i}, (a_{N_k}+d_{N_k})^{1/2}M^{-i-1}] \qquad i=0, 1, \cdots, L.$$

We observe that

$$a_{N_{k-1}}+d_{N_{k-1}}=x_1(i)y_1(i)<x_2(i)y_2(i)=a_{N_k}+d_{N_k},$$
$$0<x_1(i)<x_2(i)\leqslant\widetilde{b}_{N_k},\ 0<y_1(i)<y_2(i)\leqslant\widetilde{b}_{N_k},\quad i=0,1,\cdots,L.$$

Hence $S_i(k)\subset\overline{D}_{N_k}-\overline{D}_{N_{k-1}}$, and $S_i(k), i=0,1,\cdots,L$, are disjoint rectangles for each $k$. By the assumption $\rho_2=1$ and imitating (1.4.56), we have

$$(1-\varepsilon)a_{N_k}\leqslant(1-\varepsilon)(a_{N_k}+d_{N_k})\leqslant\lambda(S_i(k))\leqslant\left(1-\frac{\varepsilon}{2}\right)(a_{N_k}+d_{N_k})\leqslant a_{N_k}$$

for large $k$ and every $i$, provided that $\theta$ is large enough. From here on this proof continues along the lines of that of $\rho_1<1$ as above,and we will get (1.4.54) again in this case.

If $\rho_1=1$ and $\varliminf_{N\to\infty} a_N/(a_N+d_N)<1$, there exists a sequence of positive integers $G=\{N''\}$ such that

$$\lim_{N''\to\infty} a_{N''}/(a_{N''}+d_{N''})=1,\quad \varlimsup_{\substack{N'\to\infty\\ N'\notin G}} a_{N'}/(a_{N'}+d_{N'})<1.$$

By merging the previous discussions for $\rho_1<1$ and $\rho_2=1$, one of the following inequalities at least holds :

$$\sum_{N_k\in G}P\{\max_{0\leqslant i\leqslant L}\beta^*(a_{N_k}+d_{N_k},a_{N_k})|W(S_i(k))|\geqslant1-\varepsilon\}=\infty$$

and

$$\sum_{N_k\notin G}P\{\max_{0\leqslant i\leqslant L}\beta^*(a_{N_k}+d_{N_k},a_{N_k})|W(S_i(k))|\geqslant1-\varepsilon\}=\infty\ .$$

Hence (1.4.54) is also true in this case. The proof of (1.4.54) is completed. Consequently, we prove (1.4.40) and (1.4.41).

3° Suppose that Conditions (1.4.42) and (1.4.43) are satisfied. We prove

$$\varliminf_{T\to\infty}\sup_{0\leqslant t\leqslant d_T}\sup_{R\in L^*_{t+a_T}(a_T)}\beta^*(a_T+d_T,a_T)|W(R)|\geqslant1\qquad\text{a.s.}\qquad(1.4.59)$$

which combines with (1.4.40) together to yield (1.4.44) and (1.4.45).

Let $L=L(N)$ be the largest integer for which we have

$$(a_N+d_N)^{L+1}/d_N^L\widetilde{b}_N\leqslant\widetilde{b}_N\qquad\text{if}\quad\rho_1<1$$

$$(a_N + d_N)^{1/2} M^{L+1} \leqslant \widetilde{b}_N \qquad \text{if } \rho_2 = 1$$

and define the rectangles

$$S_i(N) = \begin{cases} [(\dfrac{d_N}{a_N + d_N})^{i+1}\widetilde{b}_N, (\dfrac{d_N}{a_N + d_N})^{i}\widetilde{b}_N] \times [0, \dfrac{(a_N + d_N)^{i+1}}{d_N^i \widetilde{b}_N}] & \text{if } \rho_1 < 1, \\ [(a_N + d_N)^{1/2} M^i, (a_N + d_N)^{1/2} M^{i+1}] \times [0, (a_N + d_N)^{1/2} M^{-i-1}] & \text{if } \rho_2 = 1, \end{cases}$$

where $i = 0, 1, \cdots, L$. We observe that

$$\lambda(S_i(N)) = \begin{cases} a_N & \text{if } \rho_1 < 1, \\ (1-\varepsilon)(a_N + d_N) & \text{if } \rho_2 = 1; \end{cases}$$

$$L \geqslant \begin{cases} c\dfrac{a_N + d_N}{a_N} \log \dfrac{\widetilde{b}_N^2}{a_N + d_N} & \text{if } \rho_1 < 1, \\ c \log \dfrac{\widetilde{b}_N^2}{a_N + d_N} & \text{if } \rho_2 = 1. \end{cases}$$

Since $S_i(N)$, $i = 0, 1, \cdots, L$, are disjoint, we have

$$\begin{aligned} &P\{ \max_{0 \leqslant i \leqslant L} \beta^*(a_N + d_N, a_N) |W(S_i(N))| \leqslant 1 - \varepsilon \} \\ &\leqslant \{ 1 - \exp(-(1-\varepsilon)(\log \frac{a_N + d_N}{a_N} + \log(\log \widetilde{b}_N / \sqrt{a_N} + 1) \\ &\qquad\qquad + \log \log(a_N + d_N)))\}^{L+1} \\ &\leqslant \exp\{-(\frac{a_N + d_N}{a_N}(\log \widetilde{b}_N/\sqrt{a_N} + 1))^{\varepsilon/2} (\log(a_N + d_N))^{-1+\varepsilon/2}\} \end{aligned}$$

provided that $N$ is large enough. Using Condition (1.4.42) and imitating the proof of (1.4.54) from (1.4.57), we get

$$\varliminf_{N \to \infty} \max_{0 \leqslant n \leqslant d_N} \sup_{R \in L^*_{n+a_N}(a_N)} \beta^*(a_N + d_N, a_N) |W(R)| \geqslant 1 - \varepsilon \quad \text{a.s.} \tag{1.4.60}$$

Notice that

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant d_T} \sup_{R \in L^*_{t+a_T}(a_T)} \beta^*(a_T + d_T, a_T) |W(R)| \qquad (1.4.61) \\ &\geqslant \max_{0 \leqslant n \leqslant d_T} \sup_{R \in L^*_{n+a_{[T]}}(a_{[T]})} \beta^*(a_T + d_T, a_T) |W(R)| \\ &\quad - \sup_{0 \leqslant t \leqslant d_T} \sup_{R \in L_{t+a_T}(a_T - a_{[T]})} \beta^*(a_T + d_T, a_T) |W(R)| \end{aligned}$$

$$=: J_1 - J_2 .$$

By recalling the proof of (1.4.52) and using the Condition (1.4.43), $\overline{\lim}_{T \to \infty} J_2 = 0$ a.s. By merging Conditions (1.4.43) and (iii$'$) we can be show

$$\beta^*(a_T + d_T, a_T) / \beta^*(a_{[T]} + d_{[T]}, a_{[T]}) \to 1 \quad \text{as } T \to \infty .$$

Hence (1.4.60) implies $\underline{\lim}_{T \to \infty} J_1 \geq 1 - \varepsilon$ a. s. Conseqńently, (1.4.59) is proved. This completes the proof of Theorem 1.4.4.

## 1.5 The Increments of a Non-Stationary Gaussian Process

Let $\{ X(t);\ t \geq 0 \}$ be a centred Gaussian process with stationary increments and $X(0) = 0$ a.s. We assume that

$$\sigma^2(h) = E(X(t+h) - X(t))^2 = EX^2(h) = C_0 h^{2\alpha} \tag{1.5.1}$$

for $0 < \alpha < 1$ and some constant $C_0 > 0$. This condition implies that the process has a continuous path with probability one (e.g. see Fernique 1964).

It is natural to ask : Are the results of the increments of a Wiener process stated in Section 1. 1. and 1. 2. also true for the Gaussian process $\{ X(t) \}$? The answer is : Yes, in many cases.

### 1.5.1 Csörgő-Révész's increments

Define

$$H(T, h) = \sup_{0 \leq t \leq T-h} \ \sup_{0 \leq s \leq h} |X(t+s) - X(t)| ,$$

$$I(T, h) = \sup_{0 \leq t < t' \leq T} \ \sup_{t' - t \leq h} |X(t') - X(t)| .$$

Ortega (1984) proved the following statements corresponding to Theorem 1.1.1.

**Theorem 1.5.1** (Ortega 1984) *Let $0<a_T\leqslant T$ be a function of $T$ which satisfy Conditions* (i) *and* (ii) *of Theorem* 1.1.1. *Then*

$$\overline{\lim_{T\to\infty}}\ \beta_T\ I(T,a_T)=\overline{\lim_{T\to\infty}}\ \beta_T\ |X(T+a_T)-X(T)|=1 \quad \text{a.s.} \tag{1.5.2}$$

*where*

$$\beta_T=\{2\sigma^2(a_T)(\log T/a_T+\log\log T)\}^{-1/2}.$$

*Furthermore, if Condition* (iii) *of Theorem* 1.1.1 *is satisfied, then*

$$\lim_{T\to\infty}\ \beta_T\ I(T,a_T)=\lim_{T\to\infty}\ \sup_{0\leqslant t\leqslant T-aT}\ \beta_T|X(t+a_T)-X(t)|=1 \quad \text{a.s.} \tag{1.5.3}$$

We will use the following three lemmas in the proof of Theorem 1.5.1.

**Lemma 1.5.1** (Fernique 1964) *Let $\{Y(t);\ t\in[0,1]\}$ be a separable, real Gaussian process with*

$$E(Y(t)-Y(s))^2\leqslant\Lambda^2(|t-s|),$$

*where $\Lambda(x)$ is continuous, non-decreasing and satisfies $\int_1^\infty\Lambda(e^{-u^2})\,du<\infty$ and also $EY^2(t)\leqslant\Gamma^2(\Gamma>0)$. Then, for $x\geqslant(4\log a)^{1/2}$, we have*

$$P\{\sup_{0\leqslant t\leqslant 1}|Y(t)|>x(\Gamma+4\int_1^\infty\Lambda(a^{-u^2})du)\}\leqslant ca^2\int_x^\infty e^{-u^2/2}\,du$$

*where $c$ is an absolute constant and $a\geqslant 2$.*

**Lemma 1.5.2** (Ortega 1984) *Let $\{X(t);\ t\geqslant 0\}$ be as in Theorem* 1.5.1. *Then, if $0<h\leqslant T$ and $z\geqslant 4$ we have*

$$P\{I(T,h)>z\sigma(h)\}\leqslant c\frac{T}{h}\ \frac{z^{5/\alpha-1}}{(\log z)^{2/\alpha}}\ e^{-z^2/2}.$$

*Proof* Let $\delta>0$ be given, $x\geqslant e$ and define $N=(2/\delta)^{1/\alpha}/h$. Then

$$P\{I(T,h)\geqslant(1+\sigma)x\delta(h)\} \tag{1.5.4}$$

$$\leqslant P\{\max_{0\leqslant i\leqslant[NT]}\ \sup_{0<s\leqslant h}|X(\frac{i}{N}+s)-X(\frac{i}{N})|>x\delta(h)\}$$

$$+P\{\max_{0\leqslant i\leqslant [NT]} \sup_{0<s\leqslant 1/N} |X(\frac{i}{N}+s)-X(\frac{i}{N})|>\delta\sigma(h)x/2\}$$

$$\leqslant (NT+1)(P\{\sup_{0\leqslant s\leqslant h}|X(s)|>x\sigma(h)\}+P\{\sup_{0\leqslant s\leqslant 1/N}|X(s)|>x\sigma(1/N)\}.$$

We shall use Fernique's inequality to obtain a bound for both probabilities. Let $\Delta>0$ and define $Y(t)=X(t\Delta)$, $0\leqslant t\leqslant 1$. Then $Y(t)$ is a centred, continuous Gaussian process with $E(Y(t)-Y(s))^2\leqslant\sigma^2(|t-s|\Delta)$ and $EY^2(t)\leqslant\sigma^2(\Delta)$. Therefore, if $v\geqslant(4\log a)^{1/2}$, we have

$$P\{\sup_{0\leqslant t\leqslant 1}|Y(t)|\geqslant v\sigma(\Delta)F(a,\Delta)\}\leqslant ca^2\int_v^\infty e^{-u^2/2}du,$$

where

$$F(a,\Delta)=1+\frac{4}{\sigma(\Delta)}\int_1^\infty\sigma(\Delta\, a^{-u^2})du\leqslant 1+\frac{2}{\alpha a^\alpha\log a}.$$

Therefore if $x=v(1+2/\alpha a^\alpha\log a)$ we get

$$P\{\sup_{0<s\leqslant\Delta}|X(s)|>x\sigma(\Delta)\}\leqslant ca^2\int_v^\infty e^{-u^2/2}du$$

$$\leqslant ca^2\psi(x)\exp\left(\frac{2\alpha x^2a^\alpha\log a+2x^2}{(2+\alpha a^\alpha\log a)^2}\right),$$

where $\psi(x)=\dfrac{1}{x\sqrt{2\pi}}e^{-x^2/2}$. Let now $a=x^{2/\alpha}/(\log x)^{1/\alpha}$, then $v\geqslant(4\log a)^{1/2}$ and $2x^2/(2+\alpha a^\alpha\log a)^2\rightarrow 0$. The remaining term in the exponent converges to 1 as $x\rightarrow\infty$ and therefore

$$P\{\sup_{0<s\leqslant\Delta}|X(s)|>x\sigma(\Delta)\}\leqslant c\frac{x^{4/\alpha}\psi(x)}{(\log x)^{2/\alpha}}.$$

Using this inequality twice in the right-hand side of (1.5.4) we obtain

$$P\{I(T,h)>(1+\delta)x\sigma(h)\}$$

$$\leqslant cNT\frac{x^{4/\alpha}}{(\log x)^{2/\alpha}}\psi(x)=c\frac{T}{h\delta^{1/\alpha}}\frac{x^{4/\alpha-1}}{(\log x)^{2/\alpha}}e^{-x^2/2}.$$

Now put $z=x(1+\delta)$, then

$$P\{I(T,h)>z\sigma(h)\}\leqslant c\frac{T}{h}\frac{z^{4/\alpha-1}}{\delta^{1/\alpha}(\log z)^{2/\alpha}}e^{-z^2/2}\exp\left(\frac{2\delta z+z\delta^2}{2(1+\delta)^2}\right).$$

Finally, choosing $\delta=1/z$, the proof is completed.

**Lemma 1.5.3** (Slepian 1962, Berman 1964) *Let $(X_j; j=1,2,\cdots,n\}$ be centred, stationary Gaussian random variables with $EX_j^2=1$ for all $j$ and $EX_iX_j=r_{ij}$. Let $I_c^{+1}=[c,\infty)$ and $I_c^{-1}=(-\infty,c)$. Denote by $F_j$ the event $\{X_j\in I_{c_j}^{\varepsilon_j}\}$ for $c_j\in R, j=1,2,\cdots,n$, where $\varepsilon_j$ is either $+1$ or $-1$. Let $K\subset\{1,2,\cdots,n\}$. Then*

(i) *$P\{\bigcap_{j\in K}F_j\}$ is an increasing function of $r_{ij}$ if $\varepsilon_i\varepsilon_j=+1$. Otherwise it is decreasing.*

(ii) *If $\{K_l, l=1,2,\cdots,s\}$ is a partition of $K$, then*

$$|P\{\bigcap_{j\in K}F_j\}-\prod_{l=1}^{s}P\{\bigcap_{j\in K_l}F_j\}|\leqslant\sum_{1\leqslant l<m\leqslant s}\ \sum_{j\in K_l}\sum_{i\in K_m}|r_{ij}|\varphi(c_i,c_j;r_{ij}^*)$$

*where $\varphi(x,y;r)$ is the standard bivariate Gaussian density with correlation $r$ and $r_{ij}^*$ is a number between 0 and $r_{ij}$.*

**Lemma 1.5.4** *Let $(A_n; n\geqslant 1\}$ be a sequence of events. If*

(i) $\sum_{n=1}^{\infty}P(A_n)=\infty$,

(ii) $\lim_{n\to\infty}\sum_{1\leqslant j<k\leqslant n}[P(A_jA_k)-P(A_j)P(A_k)]/(\sum_{j=1}^{n}P(A_j))^2=0$,

*then* $P(A_n\ \text{i.o.})=1$.

The proof of Lemma 1.5.4 can be found in Billingsley (1986).

*Proof of Theorem* 1.5.1 1° Using Lemma 1.5.2 and imitating the proof of Theorem 1.1.1 (see Csörgő and Révész 1981). one can also show that

$$\overline{\lim_{T\to\infty}}\ \beta_T I(T,a_T)\leqslant 1+\varepsilon\qquad\text{a.s.}\tag{1.5.5}$$

2° We prove

$$\overline{\lim_{T \to \infty}} \ \beta_T |X(T) - X(T-a_T)| \geqslant 1 \qquad \text{a.s.} \tag{1.5.6}$$

Let $\rho = \lim_{T \to \infty} a_T / T$. If $\rho = 1$ then necessarily $a_T = T$ and $|X(T) - X(T-a_T)| = |X(T)|$. By the law of the iterated logarithm (see Orey 1971), (1.5.6) is true. Suppose $\rho < 1$, define $T_1 = 1$, $T_k - a_{Tk} = T_{k-1}$ for $k > 1$, and $Y_k = (X(T_k) - X(T_k - a_{Tk}))/\sigma(a_{Tk})$. For any given $\varepsilon > 0$, let

$$A_n = \{\beta_{T_n} Y_n > (1-\varepsilon)/\sigma(a_{T_n})\}.$$

Imitating the proof of Theorem 1.1.1, we have that $\sum_{n=1}^{\infty} P(A_n) = \infty$. Hence, in order to prove (1.5.6), we need only show that (ii) of Lemma 1.5.4 holds.

By Lemma 1.5.3, if $EY_jY_k \leqslant 0$ then $P(A_jA_k) \leqslant P(A_j)P(A_k)$ and (ii) obviously holds; this is true if $0 < \alpha \leqslant 1/2$ in (1.5.1). Therefore we only have to consider the case $1/2 < \alpha < 1$. Suppose $k \geqslant j+3$. We have

$$EY_j Y_k = \frac{1}{2Q^\alpha R^\alpha} (P^{2\alpha} + G(Q,R))$$

where

$$P = \sum_{i=j+1}^{k-1} a_{T_i}, \ Q = a_{T_j}, \ R = a_{Tk}, G(U,V) = (P+U+V)^{2\alpha} - (P+U)^{2\alpha} - (P+V)^{2\alpha}.$$

By Taylor's theorem

$$G(Q,R) = -P^{2\alpha} + 2\alpha(2\alpha-1)P^{2\alpha-2}QR + S,$$

where

$$S = \frac{1}{3!} 2\alpha(2\alpha-1)(2\alpha-2)((Q+R)^3(P+\tau Q+\tau R)^{2\alpha-3}$$
$$-Q^3(P+\tau Q)^{2\alpha-3} - R^3(P+\tau R)^{2\alpha-3})$$

for some $0 < \tau < 1$. It is easy to see that

$$S \leqslant \frac{2\alpha(2\alpha-1)(2-2\alpha)}{3!} Q^3(P+\tau Q)^{2\alpha-3}$$

and

$$G(Q,R) \leqslant -P^{2\alpha} + 3\alpha(2\alpha-1)QRP^{2\alpha-2},$$

whence

$$EY_jY_k \leqslant c(a_{T_j}a_{T_k})^{1-\alpha}\cdot\Big(\sum_{i=j+1}^{k-1} a_{T_i}\Big)^{2(\alpha-1)}. \tag{1.5.7}$$

Since $\rho<1$, we may assume, without loss of generality, that $a_1<1$, and then $T_k(1-a_1)\leqslant T_{k-1}$ by $a_1\geqslant a_{Tk}/T_k$. So $a_{Tk}\leqslant(1-a_1)^{-1}a_{Tk-1}$ and

$$EY_jY_k\leqslant c\Big(a_{T_j}/\sum_{i=j+1}^{k-1} a_{T_i}\Big)^{1-\alpha}\leqslant c(k-j-1)^{\alpha-1}=:\eta_{jk}$$

as long as $k\geqslant j+3$.

We can now turn to the proof of (ii) of Lemma 1.5.4. Lemma 1.5.3 shows that $P(A_jA_k)-P(A_j)P(A_k)\leqslant r_{jk}\,\varphi(\lambda_j,\lambda_k;r_{jk}^*)$ with $r_{jk}=EY_jY_k$, $\lambda_k=(1-\varepsilon)\beta_{Tk}^{-1}/\sigma(a_{Tk})$. For our purpose it is enough to consider, for some fixed $m$.

$$\begin{aligned}&\sum_{k=m}^{n}\sum_{j=1}^{k-3}(P(A_jA_k)-P(A_j)P(A_k))\\ &\leqslant\left(\sum_{k=m}^{n}\sum_{j=1}^{k-\gamma_k}+\sum_{k=m}^{n}\sum_{j=k-\gamma_k}^{k-3}\right)r_{jk}\,\varphi(\lambda_j,\lambda_k;r_{jk}^*)\end{aligned}$$

where $\gamma_k=[\lambda_k^{4/(1-\alpha)}]$. We start with

$$\begin{aligned}&\sum_{k=m}^{n}\sum_{j=1}^{k-\gamma_k}\frac{r_{jk}}{2\pi(1-r_{jk}^{*2})^{1/2}}\exp\Big\{-\frac{\lambda_j^2+\lambda_k^2-2\lambda_j\lambda_kr_{jk}^*}{2(1-r_{jk}^{*2})}\Big\}\\
&\leqslant\sum_{k=m}^{n}\sum_{j=1}^{k-\gamma_k}\frac{\eta_{jk}\lambda_k\lambda_j}{(1-\eta_{jk}^2)^{1/2}}\psi(\lambda_j)\psi(\lambda_k)\exp\left\{-\frac{r_{jk}^{*2}(\lambda_j^2+\lambda_k^2)-2\lambda_j\lambda_kr_{jk}^*}{2(1-r_{jk}^{*2})}\right\}\\
&\leqslant\sum_{k=m}^{n}\sum_{j=1}^{k-\gamma_k}\frac{\eta_{jk}\lambda_k^2}{(1-\eta_{jk}^2)^{1/2}}\psi(\lambda_j)\psi(\lambda_k)\exp\{\eta_{jk}\lambda_k^2\}\end{aligned}$$

but $\eta_{jk}\lambda_k^2\leqslant c\gamma_k^{\alpha-1}\lambda_k^2\leqslant c\lambda_k^{-2}\downarrow 0$ where $j\leqslant k-\gamma_k$. Let $\delta>0$, then, choosing $m$ appropriately (but fixed) the sum we are considering is

$$\leqslant\delta\sum_{k=m}^{n}\sum_{j=1}^{k-\gamma_k}P(A_k)P(A_j)\leqslant\delta\Big(\sum_{k=1}^{n}P(A_k)\Big)^2. \tag{1.5.8}$$

The second sum is

$$\begin{aligned}&\leqslant\sum_{k=m}^{n}\sum_{j=k-\gamma_k}^{k-3}\frac{r_{jk}\lambda_j}{\sqrt{2\pi(1-r_{jk}^{*2})}}\psi(\lambda_j)\exp\Big\{-\frac{(\lambda_k-r_{jk}^*\lambda_j)^2}{2(1-r_{jk}^{*2})}\Big\}\\
&\leqslant\sum_{k=m}^{n}\sum_{j=k-\gamma_k}^{k-3}\frac{\lambda_j}{(1-r^2)^{1/2}}\psi(\lambda_j)\exp\Big\{-\frac{\lambda_k^2}{2}\Big(\frac{1-r}{1+r}\Big)\Big\}\end{aligned}$$

$$\leqslant \sum_{k=m}^{n} \sum_{j=k-\gamma_k}^{k-3} \frac{\lambda_j \psi(\lambda_j)}{(1-r^2)^{1/2}} \exp(-b\lambda_k^2)$$

where $r<1$ is the maximum of the covariances $r_{jk}$ and $b=(1-r)/2(1+r)$.

Let us consider first the sum over the indices $k$ in the set $A=\{k: m\leqslant k\leqslant n, \lambda_k \geqslant ((2/b)\log k)^{1/2}\}$. Then

$$\sum_{k\in A} \sum_{j=k-\gamma_k}^{k-3} \frac{\lambda_j \psi(\lambda_j)}{(1-r^2)^{1/2}} \exp(-b\lambda_k^2) \tag{1.5.9}$$

$$\leqslant c \sum_{k\in A} \frac{(\log k)^{1/2}}{k^2} \sum_{j=1}^{n} P(A_j) \leqslant c \sum_{j=1}^{n} P(A_j).$$

If $k\in A'=\{k: m\leqslant k\leqslant n, \lambda_k<((2/b)\log k)^{1/2}\}$, then $\gamma_k<((2/b)\log k)^{2/(1-\alpha)}$ and if $j=k-\gamma_k$ then, for some $D>0$, $k<j+D(\log j)^{2/(1-\alpha)}=j+\xi_j$ say. Changing the order of summation,

$$\sum_{k\in A'} \sum_{j=k-\gamma_k}^{k-3} \frac{\lambda_j \psi(\lambda_j)}{(1-r^2)^{1/2}} \exp(-b\lambda_k^2)$$

$$\leqslant c \sum_{j=m-\gamma_m}^{n-1} \sum_{k=j+1}^{j+\xi_j} \lambda_j \psi(\lambda_j) \exp(-b\lambda_k^2) \tag{1.5.10}$$

$$\leqslant c \sum_{j=m-\gamma_m}^{n-1} \xi_j \lambda_j \psi(\lambda_j) \exp(-b\lambda_j^2)$$

$$\leqslant c \sum_{j=1}^{n} P(A_j).$$

Thus, by using (1.5.8), (1.5.9) and (1.5.10), for any given $\delta$ there is an $m$ such that

$$\sum_{k=m}^{n} \sum_{j=1}^{k-3} (P(A_j A_k) - P(A_j)P(A_k)) \leqslant \delta\Big(\sum_{j=1}^{n} P(A_j)\Big)^2 + c\sum_{j=1}^{n} P(A_j),$$

which implies (ii) of Lemma 1.5.4 in the case $\rho<1$. Therefore, (1.5.6) holds true, and (1.5.2) is proved.

3° Suppose that Condition (iii) is satisfied. Let

$C(T)=\beta_T \lim\limits_{0\leqslant t\leqslant T-a_T} |X(t+a_T)-X(t)|$. We show that

$$\varliminf_{T\to\infty} C(T)\geqslant 1 \quad \text{a.s.} \tag{1.5.11}$$

For some $\delta>0$, define $T_n=(1+\delta)^n$, $\zeta(T)=[T/a_T]-1$ and

$$V(k,n)=(X((k+1)a_{T_n})-X(ka_{T_n}))/\sigma(a_{T_n})$$

for $0\leqslant k\leqslant\zeta(T_n)$, $n\geqslant 1$. The variables $V(k,n)$ have $EV(k,n)=0$, $EV^2(k,n)=1$ for all $(k,n)$, and it can be shown as in 2° that if $k\geqslant j+1$,

$$r_n(k,j):=EV(k,n)V(j,n)\leqslant c(k-j)^{2(\alpha-1)}$$

and is negative if $\alpha\leqslant 1/2$.

If $\alpha\leqslant 1/2$, using Lemma 1.5.3 and imitating the proof of Theorem 1.1.1, we have

$$\sum_{n=1}^{\infty} P\{\max_{0\leqslant k\leqslant\zeta(T_n)}|V(k,n)|\leqslant\lambda_n\}<\infty .$$

If $\alpha>1/2$, consider an extra term coming from (ii) of Lemma 1.5.3. Suppose that $\varepsilon$ is small enough so that $\varepsilon<1-\alpha$ and $2(1-\varepsilon)^2>1+r$, where $r=\max\{r_n(k,j):n\geqslant 1, 1\leqslant k<j\leqslant\zeta(T_n)\}$. Then

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\sum_{j=k+1}^{\zeta(T_n)} r_n(k,j)\varphi(\lambda_n,\lambda_n;r_n^*(k,j))$$
$$=\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\left(\sum_{j=k+1}^{k+\mu_n-1}+\sum_{j=k+\mu_n}^{\zeta(T_n)}\right)\frac{r_n(k,j)}{(1-r_n^2(k,j))^{1/2}}\exp\left\{-\frac{\lambda_n^2}{1+r_n^*(k,j)}\right\}$$

where $\mu_n=\lambda_n^{1/(1-\alpha)}$. Let $v>0$ be defined by $1+v=2(1-\varepsilon)^2/(1+r)$, then the first sum is

$$\leqslant c\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\sum_{j=k+1}^{k+\mu_n}\exp\{-(1+v)(\log T_n/a_{T_n}+\log\log T_n)\}$$

$$\leqslant c\sum_{n=1}^{\infty}\frac{T_n}{a_{T_n}}\mu_n\left(\frac{a_{T_n}}{T_n\log T_n}\right)^{1+v}<\infty .$$

The second sum does not exceed

$$c\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\sum_{j=k+\mu_n}^{\zeta(T_n)}\frac{1}{(j-k)^{2(1-\alpha)}}\exp\{-\lambda_n^2+c\lambda_n^2/\mu_n^{2(1-\alpha)}\}$$

$$\leqslant c\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\left(\frac{a_{T_n}}{T_n\log T_n}\right)^{2(1-\varepsilon)}\sum_{j=k+\mu_n}^{\zeta(T_n)}\frac{1}{(j-k)^{2(1-\alpha)}}$$

$$\leqslant c\sum_{n=1}^{\infty}\zeta^{2\alpha}(T_n)\left(\frac{a_{T_n}}{T_n\log T_n}\right)^{2(1-\varepsilon)}<\infty .$$

By the Borel-Cantelli lemma, we have

$$\varliminf_{n\to\infty} \lambda_n^{-1} \max_{0\leqslant k\leqslant \zeta(T_n)} |V(k,n)| \geqslant 1 \qquad \text{a.s.}$$

Suppose now $T_n \leqslant T < T_{n+1}$, then $0 \leqslant a_T - a_{T_n} \leqslant \delta a_T$ and therefore

$$\begin{aligned} C(T) &= \beta_T \sup_{0\leqslant t\leqslant T-a_T} |X(t+a_T)-X(t)| \\ &\geqslant \beta_{T_{n+1}} \max_{0\leqslant k\leqslant \zeta(T_n)} |V(k,n)|\sigma(a_{T_n}) - \beta_T \sup_{0\leqslant t\leqslant T-\delta a_T} \sup_{0\leqslant s\leqslant \delta a_T} |X(t+s)-X(t)|, \end{aligned}$$

by using 1° and the fact that $\beta_{T_{n+1}}/\beta_{T_n}$ is arbitrarily close to one if $\delta$ is small enough. The proof is completed.

*Remark* 1.5.1 Hong (1990) discussed the inferior limit of the increments of the Gaussian process $\{X(t)\}$ and obtained the same results as Book and Shore (1978). The modulus of continuity of the process $\{X(t)\}$ was given by Lu (1986).

The lag increments of the process $\{X(t)\}$ have been discussed by Lu (1986) and Hong (1990).

**Theorem 1.5.2** Let $\{X(t);\ t\geqslant 0\}$ be as above, for which (1.5.1) is satisfied, then

$$\varlimsup_{T\to\infty} \sup_{0<t\leqslant T} |X(T)-X(T-t)|/d(T,t) = 1 \qquad \text{a.s.} \tag{1.5.12}$$

$$\lim_{T\to\infty} \sup_{0<t\leqslant T} \sup_{t\leqslant s\leqslant T} |X(s)-X(s-t)|/d(T,t) = 1 \qquad \text{a.s.} \tag{1.5.13}$$

$$\varlimsup_{T\to\infty} \sup_{0<t\leqslant T} \sup_{0\leqslant s\leqslant t} |X(T)-X(T-s)|/d(T,t) = 1 \qquad \text{a.s.} \tag{1.5.14}$$

where $d(T,t) = \{2\sigma^2(t)(\log T/t + \log\log t)\}^{1/2}$.

The proof is analogous to that of Theorem 1.1.3 and Theorem 1.5.1. So we omit it here.

*Remark* 1.5.2 The conclusions that correspond to (1.1.44), (1.1.45), (1.1.72), (1.1.73) and (1.2.1), (1.2.2) are also true for the Gaussian process $\{X(t)\}$.

### 1.5.2 Further discussion for increments of a Gaussion process

Let $\{ \Gamma(t); -\infty < t < \infty \}$ be an almost surely continuous mean zero Gaussian process with stationary increments. We assume that

$$\sigma^2(s) = E(\Gamma(t+s) - \Gamma(t))^2 \tag{1.5.15}$$

is a monotone non-decreasing function of $s$ and $\sigma(s) = s^{\alpha}\sigma_1(s), s>0$, for some $\alpha>0$, where $\sigma_1(s)$ is a non-decreasing function. The large increment results for the Gaussian process $\{\Gamma(t); t\geqslant 0\}$ were obtained by Csáki, Csörgő, Lin and Révész (1990).

**Theorem 1.5.3** (Csáki et al. 1990) *Let* $0<a_T\leqslant T$ *be a function of* $T$. *If the Conditions* (i) *and* (ii) *of Theorem* 1.1.1 *are satisfied, in addition, for any* $0\leqslant a<b\leqslant c<d$ *we have also*

$$E(\Gamma(b)-\Gamma(a))(\Gamma(d)-\Gamma(c))\leqslant 0, \tag{1.5.16}$$

*then we have*

$$\overline{\lim_{T\to\infty}} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} \beta_T|\Gamma(t+s)-\Gamma(t)|=1 \quad \text{a.s.} \tag{1.5.17}$$

$$\overline{\lim_{T\to\infty}} \sup_{0\leqslant t\leqslant T-a_T} \beta_T|\Gamma(t+a_T)-\Gamma(t)|=1 \quad \text{a.s.} \tag{1.5.18}$$

$$\overline{\lim_{T\to\infty}} \sup_{0\leqslant s\leqslant a_T} \beta_T|\Gamma(T+s)-\Gamma(T)|=1 \quad \text{a.s.} \tag{1.5.19}$$

$$\overline{\lim_{T\to\infty}} \beta_T|\Gamma(T+a_T)-\Gamma(T)|=1 \quad \text{a.s.} \tag{1.5.20}$$

*where* $\beta_T=(2\sigma^2(a_T)(\log(T/a_T)+\log\log T))^{-1/2}$.

If Condition (iii) of Theorem 1.1.1 is also satisfied, then $\overline{\lim}$ can be replaced by lim in (1.5.17) and (1.5.18).

The proof of Theorem 1.5.3 is similar to that of Theorem 1.2.1 in Csörgő-Révész (1981), here Lemma 1.5.4 is used with the help of Condition (1.5.16). The details are omitted.

Chapter 2

# The Increments of Partial Sums of Independent Random Variables

## 2.1 Introduction

The almost sure limiting behavior of the increments of partial sums of random variables is one of the profound results in the limit theorems of probability theory. For an i.i.d. sequence, some nice conclusions have been obtained with the help of strong approximations of partial sums by a Wiener process and together with the increments of the same process.

Let $\{X_n; n \geqslant 1\}$ be a sequence of i.i.d. random variables with mean zerro and variance one, and $\{a_n; n \geqslant 1\}$ be a monotonically non-decreasing sequence of integers satisfying the conditions:

(i) $1 \leqslant a_n \leqslant n$,

(ii) $n/a_n$ is monotonically non-decreasing.

Put $S_n = \sum_{i=1}^{n} X_i$, $\beta_N = \{2a_N(\log(N/a_N) + \log\log N)\}^{-1/2}$. At first, Csörgő and Révész (1981) investigated large increments of $\{S_n\}$ and proved

**Theorem 2.1.1** *Suppose that* $\{X_n\}$ *satisfies the condition*

$$\text{there exists a } t_0 > 0 \text{ such that } Ee^{tX_1} \text{ is finite for } |t| < t_0. \tag{2.1.1}$$

*Suppose that in addition to Conditions* (i) *and* (ii), $\{a_n\}$ *satisfies also*

(iii) $a_n/\log n \to \infty$ *as* $n \to \infty$.

*Then we have*

$$\overline{\lim_{N \to \infty}} \max_{1 \leqslant n \leqslant N - a_N} \max_{1 \leqslant k \leqslant a_N} \beta_N |S_{n+k} - S_n| = 1 \quad \text{a.s.} \tag{2.1.2}$$

$$\overline{\lim_{N \to \infty}} \max_{1 \leqslant n \leqslant N - a_N} \beta_N |S_{n+a_N} - S_n| = 1 \quad \text{a.s.} \tag{2.1.3}$$

$$\overline{\lim_{N \to \infty}} \max_{1 \leqslant k \leqslant a_N} \beta_N | S_{N+k} - S_N | = 1 \quad \text{a.s.} \tag{2.1.4}$$

$$\overline{\lim_{N \to \infty}} \beta_N | S_{N+a_N} - S_N | = 1 \quad \text{a.s.} \tag{2.1.5}$$

*If we have also*

(iv) $\lim_{n \to \infty} (\log n/a_n)/\log \log n = \infty$,

*then*

$$\lim_{N \to \infty} \max_{1 \leqslant n \leqslant N-a_N} \max_{1 \leqslant k \leqslant a_N} \beta_N | S_{n+k} - S_n | = 1 \quad \text{a.s.} \tag{2.1.6}$$

$$\lim_{N \to \infty} \max_{1 \leqslant n \leqslant N-a_N} \beta_N | S_{n+a_N} - S_n | = 1 \quad \text{a.s.} \tag{2.1.7}$$

If we assume only the existence of a finite number of moments instead of that of the moment generating function, Csörgő and Révész also proved.

**Theorem 2.1.2** *Let* $H(x) > 0$, $x > 0$, *be a non-decreasing continuous function for which the following assumptions hold*:

$$EH(|X_1|) < \infty, \tag{2.1.8}$$

$$\underline{\lim}_{x \to \infty} H(\varepsilon x)/H(x) > 0 \quad \text{for any} \quad \varepsilon > 0, \tag{2.1.9}$$

$$x^{-(2+\varepsilon)}H(x) \text{ is an increasing function of } x \text{ for some } \varepsilon > 0, \tag{2.1.10}$$

$$x^{-1}\log H(x) \text{ is non-increasing.} \tag{2.1.11}$$

In addition to Conditions (i) and (ii), $\{a_n\}$ also satisfies that there exists a $C > 0$ such that

$$a_n \geqslant C (\mathrm{inv} H(n))^2 / \log n. \tag{2.1.12}$$

Then the conclusions of Theorem 2.1.1 are true.

Later, Hanson and Russo (1983) considered another form of sums of i.i.d. random variables, the so-called lag sums. Put $d(N, k) = \{2k(\log(N/k) + \log \log k)\}^{1/2}$. They proved

**Theorem 2.1.3** *Suppose that* $\{X_n\}$ *satisfies the condition of Theorem* 2.1.1. *Let* $\{a_n\}$ *be a sequence of integers satisfying Conditions* (i) *and* (iii). *Then we have*

$$\overline{\lim_{N \to \infty}} \max_{a_N \leqslant k \leqslant N} | S_N - S_{N-k} | / d(N, k) = 1 \quad \text{a.s.} \tag{2.1.13}$$

$$\overline{\lim_{N\to\infty}} \max_{a_N\leqslant k\leqslant N} \max_{1\leqslant j\leqslant k} |S_N - S_{N-j}|/d(N,k) = 1 \quad \text{a.s.} \tag{2.1.14}$$

$$\overline{\lim_{N\to\infty}} \max_{0\leqslant m\leqslant n\leqslant N} \max_{a_N\leqslant n-m} |S_n - S_m|/d(n, n-m) = 1 \quad \text{a.s.} \tag{2.1.15}$$

$$\overline{\lim_{N\to\infty}} \max_{0\leqslant m\leqslant j\leqslant k\leqslant n\leqslant N} \max_{a_N\leqslant n-m} |S_k - S_j|/d(n, n-m) = 1 \quad \text{a.s.} \tag{2.1.16}$$

*If conditions* (2.1.1) *and* (iii) *are respectively replaced by*

$$\text{there exists } r > 2 \text{ such that } E|X_1|^r < \infty, \tag{2.1.17}$$

$$\underline{\lim_{n\to\infty}} a_n (\log n)/n^{2/r} > 0, \tag{2.1.18}$$

*then the conclusions of the theorem remain true.*

Furthermore, Csörgő and Révész (1981) considered also how small the increments of partial sums of i.i.d. random variables are. They gave the following theorem by applying a small deviation theorem of Moguls'kii (1974).

**Theorem 2.1.4** *Let* $\{X_n\}$ *be a sequence of* i.i.d. *random variables with mean* 0 *and variance* 1. *Let* $\{a_n\}$ *be a non-decreasing sequence of integers satisfying Conditions* (i), (ii) *and* (iii). *Then we have*

$$\underline{\lim_{N\to\infty}} \min_{1\leqslant n\leqslant N-a_N} \max_{1\leqslant k\leqslant a_N} \gamma_N |S_{n+k} - S_n| = 1 \quad \text{a.s.} \tag{2.1.19}$$

where $\gamma_N = \left( \frac{8}{\pi^2} \frac{\log N/a_N + \log\log N}{a_N} \right)^{1/2}$. If Condition (iv) is added, then

$$\lim_{N\to\infty} \min_{1\leqslant n\leqslant N-a_N} \max_{1\leqslant k\leqslant a_N} \gamma_N |S_{n+k} - S_n| = 1 \quad \text{a.s.} \tag{2.1.20}$$

They declare that this theorem can be proved by repeating the proof of the corresponding theorem for a Wiener process (Theorem 1.7.1 of Csörgő and Révész 1981). Unfortunately, this seems to be impossible from a careful investigation of that proof (check the proof from (1.7.4) to (1.7.5) on pages 49 — 50 in that book).

In this chapter we intend to give that a.s. limiting behavior of increments of partial sums of independent but not necessarily identically-distributed random variables. At this time, the trouble is that we have not yet had the

perfect result of strong approximations as apply for i. i. d. random variables. Lin (1986a, 1987, 1988b) obtained results corresponding to Theorem 2.1.1. and 2.1.2 by a direct approach. At the same time, Condition (ii) on $\{ a_n \}$ is weakened essentially.

Hanson and Russo (1985) generalized the results on lag sums of i. i. d. random variables to the case of independent but not necessarily identically distributed random variables. Lin (1988a) improved their main conclusions making them correspond to that for i. i. d. case.

Recently, Shao (1989) extended Lin's results further to the more general case by the Skorohod embedding scheme.

In Sections 2. 2 and 2. 3, we state the above-mentioned results. All of these conclusions for not only the non-i. i. d. case but also the i. i. d. case are drawn under moment (or moment generating function) conditions. But strong limit theorems depend (in principle) on probabilities rather than moments. Lin (1990b) discussed the big increments of partial sums of independent random variables without moment hypotheses. His theorem is a generalization of the results with moment conditions. Recently, Lin and Shao (1990) weakened the conditions of this theorem. Section 2. 4 is addressed to this problem.

In Section 2.5, we describe a theorem, which is due to Shao (1989), concerning small increments. It not only revises the proof in Csörgő and Révész (1981), but also generalizes the result to the case of not identical-distributed random variables.

Sakhanenko (1984) established a strong approximation theorem on independent non-identically distributed random variables via a development of the method of Komlós, Major and Tusnády (1975, 1976). Shao (1989) further extended Sakhanenko's result, and then drew some conclusions on the increments of a sequence of independent non-identically distributed random variables as consequences of strong approximation theorems. In the last section, we will state the results concerned, which are different from those in Sections 2.2 and 2.3.

## 2.2 How Large Are the Lag Sums?

The motivation for the investigation of lag sums came from a specific statistical problem. When one estimates a mean by a sample $X_1, \cdots, X_n$, there will usually be bias associated with the earlier $Xk$'s . One might hope to reduce this bias by discarding some of the earlier $Xk$'s . Hanson and Russo (1983a) first obtained a.s. limit results for lag sums of i.i.d. random variables by proving limit properties for corresponding increments of a Wiener process (cf. Section 1.1.2). They thew (1985) generalized the results to the case of independent but not necessarily identically-distributed random variables. The conclusions for the i.i.d. case are close to ideal, but those for the noni.i.d. case are not. Lin (1988 a) improved the and obtained the results corresponding to those for the i.i.d. case.

Let $\{X_n; n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = 0$ ($n \geqslant 1$). Put $S_n = \sum_{i=1}^{n} X_i$, $\sigma_n^2 = EX_n^2$, $\sigma_{nk}^2 = \sum_{i=n-k+1}^{n} \sigma_i^2$ and $g(N, k) = \sigma_{Nk}\{2(\log N/k + \log\log k)\}^{1/2}$.

**Theorem 2.2.1** (Lin 1988a) *Suppose that for* $\{X_n\}$

(i) $\varliminf_{n \to \infty} \inf_{m \geqslant 0} \sum_{i=m+1}^{m+n} \sigma_i^2/n > 0$;

(ii) there exists $r > 2$ such that for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P(|X_n|^r > \varepsilon n) < \infty$$

*and for any* $s < r$ *and every* $n$,

$$E|X_n|^s \leqslant M < \infty .$$

*Then for any* $d > 0$

$$\varlimsup_{N \to \infty} \max_{dN^{2/r}/\log N \leqslant k \leqslant N} |S_N - S_{N-k}|/g(N,k) = 1 \quad \text{a.s.} \tag{2.2.1}$$

$$\overline{\lim_{N \to \infty}} \max_{dN^{2/r}/\log N \leqslant k \leqslant N} \max_{1 \leqslant j \leqslant k} |S_N - S_{N-j}|/g(N,k) = 1 \quad \text{a.s.} \quad . \tag{2.2.2}$$

We will need the following lemma for the proof of the theorem.

**Lemma 2.2.1** (a) *Suppose that there exist positive constants* $g_1, \cdots, g_n$ *and* $T$ *such that*

$$Ee^{tx_k} \leqslant e^{g_k t^2/2} \quad k = 1, 2, \cdots, n$$

*for* $0 \leqslant t \leqslant T$ *. Then, putting* $G = \sum_{k=1}^{n} g_k$, we have

$$P\{\max_{1 \leqslant k \leqslant n} S_k \geqslant x\} \leqslant e^{-x^2/2G} \quad \text{for } 0 \leqslant x \leqslant GT, \tag{2.2.3}$$

$$P\{\max_{1 \leqslant k \leqslant n} S_k \geqslant x\} \leqslant e^{-Tx/2} \quad \text{for } x \geqslant GT. \tag{2.2.4}$$

(b) *Suppose that there exists* $d > 0$ *such that*

$$|X_k| \leqslant ds_n$$

*for* $1 \leqslant k \leqslant n$ *and* $n \geqslant 1$, *where* $s_n^2 = \sum_{k=1}^{n} \sigma_k^2$. *If* $\varepsilon > 0$, *there exist constants* $\gamma(\varepsilon)$ *and* $\pi(\varepsilon)$ *such that when* $\gamma \geqslant \gamma(\varepsilon)$ *and* $\gamma d \leqslant \pi(\varepsilon)$, *we have*

$$P\{S_n \geqslant \gamma s_n| \geqslant e^{-(1+\varepsilon)\gamma^2/2}. \tag{2.2.5}$$

The conclusions (a) and (b) can be found from Petrov (1975) and Stout (1974), respectively.

Proof of Theorem 2.2.1.

Put $\beta = 1/2 + 1/r + \varepsilon$ for $0 < \varepsilon < 1/2 - 1/r$. Choose $\alpha > 0$ so that $\eta := 2\alpha(r - 2 + \frac{1}{r} - \alpha) < \frac{1}{2} - \frac{1}{r}$ and choose $m$ and $\frac{2}{r} + \eta := \beta_0 < \beta_1 < \cdots < \beta_m = \beta$ so that $(1-\beta_i)/(1-\beta_{i-2}) > (1 + \frac{\varepsilon}{2})^{-1}$ for $i = 1, \cdots, m$, where $\beta_{-1} = \beta_0 - \frac{\varepsilon}{4}$. We have $\beta_i - \beta_{i-1} < \varepsilon/2$. Put $N(i) = [N^{\beta_i}]$. Write

$$\max_{dN^{2/r}/\log N \leqslant k \leqslant N^{\beta}} = \max_{dN^{2/r}/\log N \leqslant k \leqslant N^{\beta_0}} \bigvee \max_{1 \leqslant i \leqslant m} \max_{N(i-1) \leqslant k \leqslant N(i)}$$

Consider $\max_{N(i-1)\le k\le N(i)}$ for each fixed $i$, $1\le i\le m$. At first, we prove that

$$\overline{\lim_{N\to\infty}} \max_{N(i-1)\le k\le N(i)} \max_{1\le j\le k} |S_N - S_{N-j}|/g(N,k) \le 1+C_0\varepsilon \quad \text{a.s.} \tag{2.2.6}$$

for some $C_0>0$.

Let $Y_n = X_n I(|X_n|\le n^{1/r-\alpha})$, $Z_n = X_n I(n^{1/r-\alpha}<|X_n|\le n^{1/r})$, $Y_n' = Y_n - EY_n$, $Z_n' = Z_n - EZ_n$, $T_n = \sum_{i=1}^n Y_i'$, $U_n = \sum_{i=1}^n Z_i'$, $\lambda_n^2 = \operatorname{Var} Y_n = \operatorname{Var} Y_n'$, $\lambda_{nk}^2 = \sum_{i=n-k+1}^{n} \lambda_i^2$.

Since

$$\sum_{n=1}^\infty P\{X_n \ne Y_n + Z_n\} = \sum_{n=1}^\infty P\{|X_n|>n^{1/r}) < \infty$$

by Condition (ii), $P\{X_n\ne Y_n+Z_n, \text{ i. o.}\}=0$ and we have

$$\text{the sums } \Sigma X_i \text{ in the theorem can be replaced by } \Sigma(Y_i+Z_i). \tag{2.2.7}$$

Furthermore, taking $s$ such that $(r+2)/2<s<r$, we have

$$|E(Y_i+Z_i)| = \left|\int_{|X_i|<i^{1/r}} X_i dp\right| \le i^{-(s-1)/r} E|X_i|^s < ci^{-1/2},$$

which implies that for $k\le N$

$$\sum_{i=N-k+1}^{N} |E(Y_i+Z_i)| \le c \sum_{i=N-k+1}^{N} i^{-1/2} \le ck^{1/2} = o(g(N,k)) \qquad \text{as } N\to\infty.$$

Hence (2.2.7) is equivalent to

$$\text{the sums } \Sigma X_i \text{ in the theorem can be replaced by } \Sigma(Y_i'+Z_i'). \tag{2.2.8}$$

We prove that

$$\overline{\lim_{N\to\infty}} \max_{N(i-1)\le k\le N(i)} \max_{1\le j\le k} |U_N - U_{N-j}|/g(N,k) \le \varepsilon \quad \text{a.s.} \tag{2.2.9}$$

It is easy to see that

$$EZ_n'^2 = E|Z_n'|^{r-\alpha} |Z_n'|^{-(r-\alpha-2)} \le cn^{-(r-\alpha-2)(1/r-\alpha)} = cn^{-1+2/r+\eta/2}.$$

Therefore

$$Ee^{tZ_n'} = 1 + \frac{t^2}{2} EZ_n'^2 + \frac{t^3}{6} EZ_n'^3 + \frac{t^4}{24} EZ_n'^4 + \cdots \tag{2.2.10}$$

$$\le 1 + \frac{t^2}{2} EZ_n'^2 \{1 + \frac{t}{3} 2n^{1/r} + \frac{t^2}{12} 2^2 n^{2/r} + \cdots\}$$

$$\leqslant 1+\frac{t^2}{2}cn^{-1+2/r+\eta/2}\exp\left\{\frac{\eta}{3}\log n\right\}\leqslant \exp\left\{\frac{t^2}{2}n^{-1+2/r+\eta}\right\}$$

for $0<t\leqslant \frac{\eta}{2}n^{-1/r}\log n$ and large $n$. Write

$$P_N:=P\left\{\max_{N(i-1)\leqslant k\leqslant N(i)}\max_{1\leqslant j\leqslant k}|U_N-U_{N-j}|/g(N,k)\geqslant\varepsilon\right\}$$

$$\leqslant P\left\{\max_{1\leqslant j\leqslant N(i)}|U_N-U_{N-j}|\geqslant\varepsilon\min_{N(i-1)\leqslant k\leqslant N(i)}g(N,k)\right\}.$$

Let the parameters in Lemma 2.2.1 be $T=\frac{\eta}{2}N^{-1/r}\log N$, $g_k=N^{-1+2/r+\eta}$, $G=N(i-1)^{1/2}N^{1/r}$(it can be verified that $N(i)g_k\leqslant G$), $x=\varepsilon\min_{N(i-1)\leqslant k\leqslant N(i)}g(N,k)\leqslant c\varepsilon N(i-1)^{1/2}\log^{1/2}N$. We have $0\leqslant x\leqslant GT$ for large $N$. It follows from (2.2.3) that

$$P_N\leqslant 2\exp\{-c\varepsilon^2N(i-1)^{1/2}N^{-1/r}\log N\}\leqslant N^{-2}$$

for all large $N$. (2.2.9) is proved. Thus (2.2.8) implies that

$$\text{the sums }\Sigma X_i\text{ in the theorem can be replaced by }\Sigma Y_i'. \tag{2.2.11}$$

Obviously, $\lambda_n/\sigma_n\to 1$ as $n\to\infty$. Hence, using Conditions (i) and (ii), we have

$$\lambda_{nk}/\sigma_{nk}\to 1\text{ uniformly in }k,\ 1\leqslant k\leqslant n,\text{ as }n\to\infty. \tag{2.2.12}$$

Imitation of (2.2.10) leads to

$$Ee^{tY_n'}\leqslant 1+\frac{t^2}{2}\lambda_n^2\exp\left\{\frac{2t}{3}n^{1/r-\alpha}\right\}\leqslant\exp\left\{\frac{t^2}{2}\lambda_n^2(1+\varepsilon)\right\}$$

for $0<t\leqslant n^{-1/r+\alpha/2}$ and large $n$. Put $N'(i)=[(\beta_i-\beta_{i-1})\log N/\log B]+1$ for some $B>1$. Write

$$\max_{N(i-1)\leqslant k\leqslant N(i)}\max_{1\leqslant j\leqslant k}|T_N-T_{N-j}|/\lambda_{Nk}\{2(\log N/k+\log\log k)\}^{1/2}$$

$$\leqslant\max_{1\leqslant l\leqslant N'(i)}\Big\{\max_{1\leqslant j\leqslant B^{l-1}N(i-1)}|T_N-T_{N-j}|/\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^lN(i-1)))^{1/2}$$

$$+\max_{B^{l-1}N(i-1)<j\leqslant B^lN(i-1)}|Y'_{N-j+1}+\cdots+Y'_{N-[B^{l-1}N(i-1)]}|$$

$$/\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^lN(i-1)))^{1/2}\Big\}. \tag{2.2.13}$$

Let $n_k=n(k,\ i)=[k^{(1-\beta_{i-2})^{-1}}]$, $n_{k,\ i}=[(1-\beta_{i-2})^{-1}(n_k+1)^{\beta_{i-2}}]$. And let parameters in Lemma 2.2.1 $T=n_k^{-1/r+\alpha/2}$, $G=(1+\varepsilon)\sum_{j=n_k-n_{k,i}}^{n_u}\lambda_j^2$, $x=(1+\varepsilon)\cdot\lambda_{n_k,n_{k,i}}(2\log n_k)^{1/2}$. We can verify $0\leqslant x\leqslant GT$ as well. Hence, by (2.2.3) we have

$$P\{\max_{1\leqslant j\leqslant n_{k,i}}|T_{n_k}-T_{n_{k-j}}|\geqslant(1+\varepsilon)\lambda_{n_k,n_{k,i}}(2\log n_k)^{1/2}\}$$
$$\leqslant 2\exp\{-(1+\varepsilon)(1-\beta_{i-2})^{-1}\log k\},$$

which implies that

$$\overline{\lim_{k\to\infty}}\max_{1\leqslant j\leqslant n_{k,i}}|T_{n_k}-T_{n_{k-j}}|/\lambda_{n_k,n_{k,i}}(2\log n_k)^{1/2}\leqslant 1+\varepsilon \quad \text{a.s.} \tag{2.2.14}$$

Similarly, let $T=n_k^{-1/r+\alpha/2}$, $G=(1+\varepsilon)\sum_{j=n_k-n(k,i,l,\varepsilon)}^{n_k}\lambda_j^2$, $x=(1+\varepsilon)\lambda_{n_k,n(k,i,l,\varepsilon)}\cdot(2\log(n_k/B^ln_k(i-1)))^{1/2}$, where $n(k,i,l,\varepsilon)=[(1+\varepsilon)B^{l-1}n_k(i-1)]$. We have $0\leqslant x\leqslant GT$. Therefore, by noting $n_k/B^ln_k(i-1)\geqslant n_k/n_k(i)\sim n_k^{1-\beta_i}\geqslant k^{(1+\varepsilon/2)^{-1}}$,

$$P\{\max_{1\leqslant l\leqslant n_k'(i)}\max_{1\leqslant j\leqslant n(k,i,l,\varepsilon)}|T_{n_k}-T_{n_{k-j}}|$$
$$\geqslant(1+\varepsilon)\lambda_{n_k,n(k,i,l,\varepsilon)}(2\log(n_k/B^ln_k(i-1)))^{1/2}\}$$
$$\leqslant\sum_{l=1}^{n_k'(i)}2\exp\{-(1+\varepsilon)(1+\frac{\varepsilon}{2})^{-1}\log k\}\leqslant c(\log k)k^{-(1+\varepsilon/3)}.$$

It follows that

$$\overline{\lim_{k\to\infty}}\max_{1\leqslant l\leqslant n_k'(i)}\max_{1\leqslant j\leqslant n(k,i,l,\varepsilon)}|T_{n_k}-T_{n_{k-j}}| \tag{2.2.15}$$
$$/\lambda_{n_k,n(k,i,l,e)}(2\log(n_k/B^ln_k(i-1)))^{1/2}\leqslant 1+\varepsilon \quad \text{a.s.}$$

For any positive integer $N$, there exists $k$ such that $n_{k-1}<N\leqslant n_k$. Write

$$\frac{|T_N-T_{N-j}|}{\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^lN(i-1)))^{1/2}} \tag{2.2.16}$$
$$\leqslant\frac{|T_{n_k}-T_N|}{\lambda_{n_k,n_{k,i}}(2\log n_k)^{1/2}}\cdot\frac{\lambda_{n_k,n_{k,i}}(2\log n_k)^{1/2}}{\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^lN(i-1)))^{1/2}}$$

$$+\frac{|T_{n_k}-T_{N-j}|}{\lambda_{n_k,n(k,i,l,\varepsilon)}(2\log(n_k/B^l n_k(i-1)))^{1/2}}$$

$$\cdot\frac{\lambda_{n_k,n(k,i,l,\varepsilon)}(2\log(n_k/B^l n_k(i-1)))^{1/2}}{\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^l N(i-1)))^{1/2}}$$

$$:=I_{11}I_{12}+I_{21}I_{22}.$$

Using the mean value theorem, we obtain

$$\begin{aligned} n_k-N<n_k-n_{k-1}&\leqslant(1-\beta_{i-2})^{-1}k^{(1-\beta_{i-2})^{-1}-1}\\ &\leqslant(1-\beta_{i-2})^{-1}(n_k+1)^{\beta_{i-2}},\end{aligned}$$

i.e.

$$n_k-N\leqslant n_{k,i}. \tag{2.2.17}$$

Thus (2.2.14) implies that $\overline{\lim}_{N\to\infty} I_{11}\leqslant 1+\varepsilon$ a.s. By using (2.2.12) and Conditions (i) and (ii), there exist $0<C_1\leqslant C_2<\infty$ such that

$$C_1 n\leqslant\sum_{i=m+1}^{n+1}\lambda_i^2\leqslant C_2 n$$

for every $m$ and $n$. Then we have $\lim_{N\to\infty} I_{12}=0$ uniformly in $l$, $1\leqslant l\leqslant N'(i)$, since

$$n_{k,i}/B^{l-1}N(i-1)\leqslant cn_k^{\beta_{i-2}}/N^{\beta_{i-1}}\to 0 \quad\text{as}\quad N\to\infty.$$

Hence

$$\lim_{N\to\infty}\max_{1\leqslant l\leqslant N'(i)} I_{11}I_{12}=0 \qquad \text{a.s.} \tag{2.2.18}$$

Consider $I_{21}$ and $I_{22}$. For $1\leqslant j\leqslant B^{l-1}n_k(i-1)$,

$$n_k-N+j\leqslant n_{k,i}+B^{l-1}n_k(i-1)\leqslant n(k,i,l,\varepsilon).$$

so, for $I_{21}$, (2.2.15) implies that

$$\overline{\lim}_{N\to\infty}\max_{1\leqslant l\leqslant N'(i)}\max_{1\leqslant j\leqslant B^{l-1}N(i-1)}|T_{n_k}-T_{N-j}| \tag{2.2.19}$$

$$/\lambda_{n_k,n(k,i,l,\varepsilon)}(2\log(n_k/B^l n_k(i-1)))^{1/2}\leqslant 1+\varepsilon \qquad \text{a.s.}$$

For $I_{22}$, putting $q=(N-[B^{l-1}N(i-1)])\wedge(n_k-n(k,i,l,\varepsilon))$, $Q=(N-[B^{l-1}\cdot N(i-1)])\vee(n_k-n(k,i,l,\varepsilon))$, we write

$$\frac{\lambda^2_{n_k,n(k,i,l,\varepsilon)}}{\lambda^2_{N,[B^{l-1}N(i-1)]}}\leqslant\frac{\lambda^2_{n_k,n_k-N}+\lambda^2_{N,[B^{l-1}N(i-1)]}+\lambda^2_{Q,Q-q}}{\lambda^2_{N,[B^{l-1}N(i-1)]}}. \tag{2.2.20}$$

From (2.2.17), $\lambda^2_{n_k,n_k-N}/\lambda^2_{N,[B^{l-1}N(i-1)]}\to 0$ as $N\to\infty$. And $Q-q\leqslant n_{k,i}+$

$\varepsilon B^{l-1}n_k(i-1)$, so that

$$\overline{\lim_{N\to\infty}}\ \lambda^2_{Q,Q-q}/\lambda^2_{N,[B^{l-1}N(i-1)]}\leqslant C'\varepsilon$$

for some $C'>0$. Inserting these into (2.2.20), we get

$$\overline{\lim_{N\to\infty}}\ \lambda^2_{n_k,n(k,i,l,\varepsilon)}/\lambda^2_{N,[B^{l-1}N(i-1)]}\leqslant 1+C'\varepsilon \tag{2.2.21}$$

uniformly in $l$. Furthermore, it is easy to see that

$$\lim_{N\to\infty}\ \log(n_k/B^l n_k(i-1))/\log(N/B^l N(i-1))=1. \tag{2.2.22}$$

Combining (2.2.21) and (2.2.22) with (2.2.19) yields that

$$\overline{\lim_{N\to\infty}}\ \max_{1\leqslant l\leqslant N'(i)}\ \max_{1\leqslant j\leqslant B^{l-1}N(i-1)}\ I_{21}\,I_{22}$$

$$\leqslant(1+\varepsilon)(1+C'\varepsilon)^{1/2}\leqslant 1+(C_0-1)\varepsilon \qquad \text{a.s.} \tag{2.2.23}$$

for some $C_0>0$. Thus, for the first part of the right-hand side of (2.2.13) we have

$$\overline{\lim_{N\to\infty}}\ \max_{1\leqslant l\leqslant N'(i)}\ \max_{1\leqslant j\leqslant B^{l-1}N(i-1)}\ |T_N-T_{N-j}|$$

$$/\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^l N(i-1)))^{1/2}\leqslant 1+(C_0-1)\varepsilon \qquad \text{a.s.} \tag{2.2.24}$$

For the second part of the right-hand side of (2.2.13), write

$$P\Big\{\max_{1\leqslant l\leqslant N'(i)}\ \max_{B^{l-1}N(i-1)<j\leqslant B^l N(i)}|Y'_{N-j+1}+\cdots+Y'_{N-[B^{l-1}N(i-1)]}|$$
$$/\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^l N(i-1)))^{1/2}\geqslant\varepsilon\Big\}$$
$$\leqslant\sum_{l=1}^{N'(i)}\ \sum_{j=[B^{l-1}N(i-1)]+1}^{[B^l N(i-1)]}P\{|Y'_{N-j+1}+\cdots+Y'_{N-[B^{l-1}N(i-1)]}|$$
$$\geqslant\varepsilon\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^l N(i-1)))^{1/2}\}.$$

Estimate the probabilities of the right-hand side. Let

$$T=(N-[B^{l-1}N(i-1)])^{-1/r+\alpha/2},\quad G=\sum_{j=N-[B^l N(i-1)]+1}^{N-[B^{l-1}N(i-1)]}\lambda_j^2,$$

$$x=\varepsilon\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^l N(i-1)))^{1/2}$$

We have $0\leqslant x\leqslant GT$ for all large $N$. Hence, from (2.2.3) we obtain

$$P\{\,|\,Y'_{N-j+1}+\cdots+Y'_{N-[B^{l-1}N(i-1)]}|\leqslant \varepsilon\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^lN(i-1)))^{1/2}\}$$

$$\leqslant 2\exp\{-\varepsilon^2\lambda^2_{N,[B^{l-1}N(i-1)]}\log(N/B^lN(i-1))/\sum_{j=N-[B^lN(i-1)]+1}^{N-[B^{l-1}N(i-1)]}\lambda_j^2\}$$

$$\leqslant 2\exp\{-\varepsilon^2\frac{C_1(1-\beta)}{C_2(B-1)}\log N\}\leqslant N^{-3}$$

provided that $B=B(\varepsilon)$ is close to one. Thus

$$P\{\max_{1\leqslant l\leqslant N'(i)}\ \max_{B^{l-1}N(i-1)<j\leqslant B^lN(i-1)}|\,Y'_{N-j+1}+\cdots+Y'_{N-[B^{l-1}N(i-1)]}|$$

$$/\lambda_{N,[B^{l-1}N(i-1)]}(2\log(N/B^lN(i-1)))^{1/2}\geqslant\varepsilon\}$$

$$\leqslant N'(i)(B-1)B^{N'(i)-1}N(i-1)N^{-3}\leqslant cN^{-2},$$

which implies that

$$\overline{\lim_{N\to\infty}}\ \max_{1\leqslant l\leqslant N'(i)}\ \max_{B^{l-1}N(i-1)<j\leqslant B^lN(i-1)}|\,Y'_{N-j+1}+\cdots+Y'_{N-[B^{l-1}N(i-1)]}| \tag{2.2.25}$$

$$/\lambda_{N,[B^{l-1}N(i-1)]}(2(\log(N/B^lN(i-1)))^{1/2}\leqslant\varepsilon\qquad\text{a.s.}$$

Combining (2.2.24) and (2.2.25) with (2.2.13) yields that

$$\overline{\lim_{N\to\infty}}\ \max_{N(i-1)\leqslant k\leqslant N(i)}\ \max_{1\leqslant j\leqslant k}|\,T_N-T_{N-j}|$$

$$/\lambda_{Nk}\{2\log N/k+\log\log k)\}^{1/2}\leqslant 1+C_0\varepsilon\qquad\text{a.s.} \tag{2.2.26}$$

Recalling (2.2.11) and (2.2.12), we obtain (2.2.6).

A similar method can be used to prove

$$\overline{\lim_{N\to\infty}}\ \max_{dN^{2/r}/\log N\leqslant k\leqslant N^{\beta_0}}\ \max_{1\leqslant j\leqslant k}|\,S_N-S_{N-j}|/g(N,k)\leqslant 1+\varepsilon\qquad\text{a.s.} \tag{2.2.27}$$

(At this time, the condition $\sum_{n=1}^{\infty}P\{\,|X_n|^r>\varepsilon n\,\}<\infty$ is required for any $\varepsilon>0$.)

We omit this proof. (2.2.6) and (2.2.27) yield that

$$\overline{\lim_{N\to\infty}}\ \max_{dN^{2/r}/\log N\leqslant k\leqslant N^{\beta}}\ \max_{1\leqslant j\leqslant k}|\,S_N-S_{N-j}|/g(N,k)\leqslant 1+(C_0+1)\varepsilon\qquad\text{a.s.} \tag{2.2.28}$$

Next, we prove that

$$\overline{\lim_{N\to\infty}} \max_{N^\beta<k\leq N} \max_{1\leq j\leq k} |S_N-S_{N-j}|/g(N,k)\leq 1 \quad \text{a.s.} \tag{2.2.29}$$

Choose $s$ so that $2/r<2/s<2/r+\varepsilon$. By Theorem 4.4 of Strassen (1965), there exists a probability space on which there exists an image of $\{X_n\}$ and a related standard Wiener process $W$ (we still use $X_n$ and $S_n$ on the new space) so that

$$S_n=W(\sigma_{nn}^2)+o(\sigma_{nn}^{1/2+1/s}\log\sigma_{nn}^2) \quad \text{a.s.}$$

From Condition (ii), we have

$$S_n=W(\sigma_{nn}^2)+o(n^{1/4+1/2s}\log n) \quad \text{a.s.}$$

Thus

$$\max_{N^\beta<k\leq N} \max_{1\leq j\leq k} |S_N-S_{N-j}|/g(N,k)$$

$$\leq \max_{N^\beta<k\leq N} \max_{1\leq j\leq k} \left| \frac{W(\sigma_{NN}^2)-W(\sigma_{N-j,N-j}^2)}{g(N,k)}\right| + o(N^{1/4+1/2s}/N^{\beta/2}) \quad \text{a.s.} \tag{2.2.30}$$

It follows from (1.1.27) of Theorem 1.1.3 that

$$\overline{\lim_{N\to\infty}} \max_{N^\beta<k\leq N} \max_{1\leq j\leq k} |W(\sigma_{NN}^2)-W(\sigma_{N-j,N-j}^2)|/d(\sigma_{NN}^2,\sigma_{Nk}^2)\leq 1 \quad \text{a.s.} \tag{2.2.31}$$

where $d(t,a)=\{2a(\log t/a+\log\log a)\}^{1/2}$. From Conditions (i) and (ii), we have

$$\frac{d(\sigma_{NN}^2,\sigma_{Nk}^2)}{g(N,k)}=\left\{\frac{\log(\sigma_{NN}^2/\sigma_{Nk}^2)+\log\log\sigma_{Nk}^2}{\log N/k+\log\log k}\right\}^{1/2}\to 1 \quad \text{as } N\to\infty \tag{2.2.32}$$

uniformly in $k$, $N^\beta<k\leq N$. Note that $\frac{1}{2}+\frac{1}{s}\leq\frac{1}{2}+\frac{1}{r}+\frac{\varepsilon}{2}<\beta$, which implies that

$$o(N^{1/4+1/2s}/N^{\beta/2})=o(1). \tag{2.2.33}$$

Putting (2.2.30)—(2.2.33) together gives (2.2.29).

In order to complete the proof of the theorem, it suffices to prove that

$$\overline{\lim_{N\to\infty}} \max_{dN^{2/r}/\log N\leq k\leq N} |S_N-S_{N-k}|/g(N,k)\geq 1-\varepsilon \quad \text{a.s.} \tag{2.2.34}$$

By noting (2.2.11) and (2.2.12), (2.2.34) is equivalent to

$$\overline{\lim_{N \to \infty}} \max_{dN^{2/r}/\log N \leqslant k \leqslant N} |T_N - T_{N-k}|/\lambda_{Nk}\{2(\log N/k + \log\log k)\}^{1/2} \geqslant 1-\varepsilon \quad \text{a.s.} \tag{2.2.35}$$

Let $n_k = [k^{r/(r-2)}]$. It is enough to show that

$$\overline{\lim_{k \to \infty}} |T_{n_k} - T_{n_k - [n_k^{2/r}]-1}| / \{\lambda_{n_k, [n_k^{2/r}]+1}(2(1-2/r)\log n_k)^{1/2}\} \geqslant 1-\varepsilon \quad \text{a.s.} \tag{2.2.36}$$

Using the mean value theorem we have

$$n_k - n_{k-1} \geqslant \frac{r}{r-2}(k-1)^{2/(r-2)} - 1 > n_k^{2/r} + 1$$

for large $k$, i. e. $n_k - [n_k^{2/r}] - 1 > n_{k-1}$. We are going to employ Lemma 2.2.1 (ii). For large $k$ and $j \leqslant n_k$, we have

$$|Y_j'| \leqslant 2n_k^{1/r-\alpha} \leqslant 2k^{(1-\alpha r)/(r-2)}.$$

Let $\delta = -\alpha r/(r-2)$. Then

$$|Y_j'| \leqslant 2k^{\delta + 1/(r-2)} \leqslant C^* k^\delta \lambda_{n_k, [n_k^{2/r}]+1}$$

for some $C^* > 0$. So, if we take $d = C^* k^\delta$, $\gamma = (1-\varepsilon)\{2(1-2/r)\log n_k\}^{1/2}$ in Lemma 2.2.1 (ii), we have

$$P\{|T_{n_k} - T_{n_k - [n_k^{2/r}]-1}| \geqslant (1-\varepsilon)\lambda_{n_k, [n_k^{2/r}]+1}(2(1-2/r)\log n_k)^{1/2}\}$$
$$\geqslant \exp\{-(1+\varepsilon)(1-\varepsilon)^2(1-2/r)\log n_k\} \geqslant k^{-(1-\varepsilon)}$$

for all large $k$. By noting the independence of events and the Borel-Cantelli lemma, (2.2.36) is proved. This completes the proof of Theorem 2.2.1.

Consider the case when there exist moment generating functions. We assume that

(iii) there exist $t_0 > 0$ and $B > 0$ such that for all $k$ and $|t| \leqslant t_0$, $Ee^{tX_k} \leqslant B$.

Let $\varphi_n$ be a sequence of positive integer-values. We have

**Theorem 2.2.2** *Suppose that Conditions* (i) *and* (iii) *are satisfied, and that* $1 \leqslant \varphi_n \leqslant n$ *and* $\varphi_n/\log n \to \infty$ *as* $n \to \infty$. *Then*

$$\overline{\lim_{N \to \infty}} \max_{\varphi_N \leqslant k \leqslant N} |S_N - S_{N-k}|/g(N,k) = 1 \quad \text{a.s.} \tag{2.2.37}$$

$$\overline{\lim_{N\to\infty}} \max_{\varphi_N \leqslant k \leqslant N} \max_{1\leqslant j\leqslant k} |S_N - S_{N-j}|/g(N,k) = 1 \qquad \text{a.s.} \qquad (2.2.38)$$

In order to prove the theorem, with the help of Theorem 2.2.1, it is enough to show

$$\overline{\lim_{N\to\infty}} \max_{\varphi_N \leqslant k \leqslant N} \max_{1\leqslant j\leqslant k} |S_N - S_{N-j}|/g(N,k) \leqslant 1 \qquad \text{a.s.}$$

The procedure of its proof is similar to that for Theorem 2.2.1. We omit it.

## 2.3 How Large Are the Csörgő-Révész Increments?

The Csörgő-Révész increment is the type of that investigated first. The purpose of this section is to generalize Theorem 2.1.1 and 2.1.2 of Csörgő-Révész (1981) for an i.i.d. sequence to independent but not necessarily identically distributed sequence and to weaken the conditions in these theorems. Lin (1986a, 1987 and 1988b) first obtained results corresponding to the i.i.d. case by a direct approach. Then Shao (1989) further improved Lin's results.

Let $\{X_n; n \geqslant 1\}$ be a sequence of independent random variables with means 0. Put $S_n = \sum_{i=1}^{n} X_i$, $\sigma_n^2 = EX_n^2$. And let $\{a_n; n\geqslant 1\}$ be a sequence of integers. Put $\sigma_{nN}^2 = \sum_{i=n+1}^{n+a_N} \sigma_i^2$, $\beta_{nN} = \{2\sigma_{nN}^2[\log(N/a_N) + \log\log N]\}^{-1/2}$.

**Theorem 2.3.1** (Lin 1987) *Suppose that* $\{X_n\}$ satisfies the following conditions:

(i) $\underline{\lim}_{n\to\infty} \inf_{m\geqslant 0} \sum_{i=m+1}^{m+n} \sigma_i^2/n > 0$ *and* $E|X_n|^{2+\alpha} \leqslant M < \infty$ *for some* $\alpha > 0$ *and every* $n$;

(ii) *there exists a non-decreasing continuous function* $H(x) > 0$, $x \geqslant 0$, *satisfying*

$$\sum_{n=1}^{\infty} P\{H(|X_n|) > \varepsilon n\} < \infty \text{ for any } \varepsilon > 0; \qquad (2.3.1)$$

$$\underline{\lim}_{x \to \infty} H(x/2)/H(x) > 0 . \tag{2.3.2}$$

*And suppose that* $\{a_n\}$ *satisfies that*

(a) $a\,(\mathrm{inv}H(n))^2/\log n \leqslant a_n \leqslant n$ *for some* $a>0$.

*Then we have*

$$\overline{\lim_{N\to\infty}}\ \beta_{NN} | S_{N+a_N} - S_N | = 1 \quad \text{a.s.} \tag{2.3.3}$$

$$\overline{\lim_{N\to\infty}}\ \max_{1\leqslant n\leqslant N} \beta_{nN} | S_{n+a_N} - S_n | = 1 \quad \text{a.s.} \tag{2.3.4}$$

$$\overline{\lim_{N\to\infty}}\ \max_{1\leqslant k\leqslant a_N} \beta_{NN} | S_{N+k} - S_N | = 1 \quad \text{a.s.} \tag{2.3.5}$$

$$\overline{\lim_{N\to\infty}}\ \max_{1\leqslant n\leqslant N}\ \max_{1\leqslant k\leqslant a_N} \beta_{nN} | S_{n+k} - S_n | = 1 \quad \text{a.s.} \tag{2.3.6}$$

*If we have also*

(b) $$\lim_{N\to\infty} (\log N/a_N)/\log\log N = \infty ,$$

*then*

$$\lim_{N\to\infty}\ \max_{1\leqslant n\leqslant N} \beta_{nN} | S_{n+a_N} - S_n | = 1 \quad \text{a.s.} \tag{2.3.7}$$

$$\lim_{N\to\infty}\ \max_{1\leqslant n\leqslant N}\ \max_{1\leqslant k\leqslant a_N} \beta_{nN} | S_{n+k} - S_n | = 1 \quad \text{a.s.} \tag{2.3.8}$$

In order to prove the theorem, we need the following lemma.

**Lemma 2.3.1** *Let* $X$ *be a random variable with* $EX=0$. *Let* $a>0$ *and* $0\leqslant\alpha\leqslant 1$. *Then for any* $t\geqslant 0$, *we have*

$$E\exp\{tX\,I(X\leqslant a)\} \leqslant \exp\{\frac{t^2}{2} EX^2 | t^{2+\alpha} e^{2ta} E|X|^{2+\alpha}\}.$$

*Proof* Note that

$$E\exp\{t\,XI(X\leqslant a)\}$$

$$= 1 + tEXI(X\leqslant a) + \frac{t^2}{2} EX^2 I(X\leqslant a) + E\{\sum_{j=3}^{\infty} \frac{t^j X^j}{j!} I(X\leqslant a)\}$$

$$\leqslant 1 + \frac{t^2}{2} EX^2 + e^{ta} \frac{(ta)^{1+\alpha}}{6} t^{2+\alpha} E|X|^{2+\alpha}$$

$$\leqslant \exp\left\{ \frac{t^2}{2} EX^2 + t^{2+\alpha} e^{2t\alpha} E|X|^{2+\alpha} \right\}$$

as desired.

Proof of Theorem 2.3.1.

At first, we prove (2.3.3)—(2.3.6). It suffices to show that

$$\overline{\lim_{N\to\infty}} \max_{1\leqslant n\leqslant N} \max_{1\leqslant k\leqslant a_N} \beta_{nN}(S_{n+k}-S_n) \leqslant 1+2\varepsilon \quad \text{a.s.} \tag{2.3.9}$$

$$\overline{\lim_{N\to\infty}} \max_{1\leqslant n\leqslant N} \max_{1\leqslant k\leqslant a_N} \beta_{nN}(S_n-S_{n+k}) \leqslant 1+2\varepsilon \quad \text{a.s.} \tag{2.3.10}$$

and

$$\overline{\lim_{N\to\infty}} \beta_{NN}(S_{N+a_N}-S_N) \geqslant 1-2\varepsilon \quad \text{a.s.} \tag{2.3.11}$$

for any $0<\varepsilon<1/4$, but fixed.

Let $A$ be a positive number specified later. Define

$$Y_n = X_n I(|X_n| \leqslant A) - EX_n I(|X_n| \leqslant A),$$

$$Y_n' = X_n I(|X_n| > A) - EX_n I(|X_n| > A),$$

$$Z_n = Y_n' I\left(Y_n' \leqslant \frac{2}{A} \operatorname{inv}H(n)\right), \quad U_n = \sum_{i=1}^n Y_i, \quad T_n = \sum_{i=1}^n Z_i.$$

From Condition (2.3.2), there is a constant $d>1$ such that

$$dH(x/2) \geqslant H(x) \text{ for any } x>0,$$

and

$$d^m H(2^{-m}x) \geqslant H(x).$$

The latter implies that

$$2^{-m}\operatorname{inv}H(n) \geqslant \operatorname{inv}H(d^{-m}n).$$

Therefore

$$\frac{1}{A}\operatorname{inv}H(n) \geqslant \operatorname{inv}H(d_A n) \tag{2.3.12}$$

for some $d_A>0$. Moreover, for any $2^m \leqslant x < 2^{m+1}$ $(m\geqslant 0)$,

$$H(x) \leqslant d^{(\log x)/\log 2} H(2^{-m}x) \leqslant x^{(\log d)/\log 2} dH(1),$$

which implies that there exist positive constants $D$, $G$ and $\gamma$ such that

$$H(x) \leqslant D + Gx^{\gamma} \quad \text{for any} \quad x > 0. \tag{2.3.13}$$

$EX_n = 0$ and (2.3.13) together imply $|EX_n I(|X_n| > A)| \leqslant A \leqslant \frac{1}{A}\operatorname{inv}H(n)$ for large $n$. Using this fact and (2.3.12) we obtain

$$P\{Y_n' > \frac{2}{A}\operatorname{inv}H(n)\} \leqslant P\{|X_n| \geqslant \frac{1}{A}\operatorname{inv}H(n)\} \leqslant P\{H(|X_n|) \geqslant d_A n\}$$

for large $n$. Consequently, by condition (2.3.1)

$$P\{Y_n' > \frac{2}{A}\operatorname{inv}H(n), \text{ i.o.}\} = 0. \tag{2.3.14}$$

Then, in order to prove (2.3.9), we need only to show that

$$\overline{\lim_{N \to \infty}} \max_{1 \leqslant n \leqslant N} \max_{1 \leqslant k \leqslant a_N} \beta_{nN} |U_{n+k} - U_n| \leqslant 1 + \varepsilon \quad \text{a.s.} \tag{2.3.15}$$

$$\overline{\lim_{N \to \infty}} \max_{1 \leqslant n \leqslant N} \max_{1 \leqslant k \leqslant a_N} \beta_{nN} (T_{n+k} - T_n) \leqslant \varepsilon \quad \text{a.s.} \tag{2.3.16}$$

We now first prove (2.3.16). By Condition (i) there exist positive constants $\sigma$ and $\sigma'$ such that

$$\sigma^2 n \leqslant \sum_{i=m+1}^{m+n} \sigma_i^2 \leqslant \sigma'^2 n \quad \text{for any } m \geqslant 0 \text{ and all large } n. \tag{2.3.17}$$

Put $A_k = \{n : 2^k \leqslant a_n < 2^{k+1}\}$, $M_k = \max\{n : n \in A_k\}$. Then we have

$$\max_{N \geqslant L} \max_{1 \leqslant n \leqslant N} \max_{1 \leqslant k \leqslant a_N} \beta_{nN} (T_{n+k} - T_n) \tag{2.3.18}$$

$$\leqslant \max_{i \geqslant \log_2 a_L} \max_{N \in A_i} \max_{1 \leqslant n \leqslant N} \max_{1 \leqslant k \leqslant a_N} \beta_{nN} (T_{n+k} - T_n)$$

$$\leqslant \max_{i \geqslant \log_2 a_L} \max_{1 \leqslant n \leqslant M_i} \max_{1 \leqslant k \leqslant 2^{i+1}} (T_{n+k} - T_n)/\sigma\{2^{i+1}\log(n(\log 2^i)/2^i)\}^{1/2}.$$

Applying the Lévy maximal inequality and Lemma 2.3.1, it follows that for large $i$,

$$P\{\max_{1 \leqslant n \leqslant M_i} \max_{1 \leqslant k \leqslant 2^{i+1}} (T_{n+k} - T_n)/\sigma\{2^{i+1}\log(n(\log 2^i)/2^i)\}^{1/2} \geqslant \varepsilon\} \tag{2.3.19}$$

$$\leqslant \sum_{j=0}^{[M_i/2^i]+1} P\{\max_{1 \leqslant k \leqslant 2^{i+1}} (T_{j2^i+k} - T_{j2^i})/\sigma(2^{i+1}\log((j+1)\log 2^i))^{1/2} \geqslant \varepsilon/4\}$$

$$\leqslant 2 \sum_{j=0}^{[M_i/2^i]+1} P\{ (T_{(j+2)2^i} - T_{j2^i})/\sigma(2^{i+1}\log((j+1)\log 2^i))^{1/2} \geqslant \varepsilon/8 \}$$

$$\leqslant 2 \sum_{j=0}^{[M_i/2^i]+1} \exp\{ -\frac{\varepsilon}{8} t\sigma(2^{i+1}\log((j+1)\log 2^i))^{1/2}$$

$$+ \sum_{l=j2^i+1}^{(j+2)2^i} (\frac{t^2}{2} EZ_l^2 + t^{2+\alpha} E|Z_l|^{2+\alpha} e^{4t\,\mathrm{inv}\,H((j+2)2^i)/A}) \}$$

for any $t \geqslant 0$, where

$$\sum_{l=j2^i+1}^{(J+2)2^i} EZ_l^2 \leqslant 2 \sum_{l=j2^i+1}^{(j+2)2^i} E|X_l|^{2+\alpha}/A^\alpha \leqslant 2^{i+2}M/A^\alpha$$

by Condition (i). Put $t = t_{ji} = \dfrac{32}{\sigma\varepsilon}\left(\dfrac{\log((j+1)\log 2^i)}{2^{i+1}}\right)^{1/2}$. We have

$$\max_{j \leqslant M_i/2^i+1} t_{ji} \to 0 \quad \text{as} \quad i \to \infty .$$

Since

$$\max_{j \leqslant M_i/2^i+2} \frac{\log(j\log 2^i)}{2^i} \leqslant \max_{j \leqslant M_i/2^i+2} \frac{2\log(M_i \log M_i)}{a_{M_i}} \to 0 \quad \text{as } i \to \infty,$$

by Condition (a) and (2.3.13). Take $A$ such that

$$A \geqslant (M(32/\sigma\varepsilon)^2/2)^{1/\alpha}$$

and

$$t_{ji}^\alpha M \exp\{4t_{ji}\,\mathrm{inv}H((j+2)2^i)/A\} \leqslant (\sigma\varepsilon/32)^2$$

for large $i$. The latter is feasible since there exists $C' > 0$ such that

$$\frac{\log((j+1)\log 2^i)}{2^i}(\mathrm{inv}H((j+1)2^i))^2 \leqslant C'(\log\frac{2^i}{\log((j+1)\log 2^i})^2$$

for $0 \leqslant j \leqslant M_i/2^i+1$ by (2. 3. 13) and Condition (a) again. Therefore, from (2.3.19)

$$P\{ \max_{1 \leqslant n \leqslant M_i} \max_{1 \leqslant k \leqslant 2^{i+1}} (T_{n+k} - T_n)/\sigma(2^{i+1}\log(n(\log 2^i)/2^i))^{1/2} \geqslant \varepsilon \}$$

$$\leqslant c \sum_{j=0}^{[M_i/2^i]+1} \exp\{ -2\log((j+1)\log 2^i)\},$$

which implies (2.3.16) by noting (2.3.18).

To prove (2.3.15), let us define $A_k=\{n:\theta^k\leqslant a_n<\theta^{k+1}\}$ for some $\theta>1$, and $M_k=\max\{n:n\in A_k\}$. Then, similarly to (2.3.18),

$$\max_{N\geqslant L}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\beta_{nN}|U_{n+k}-U_n|$$

$$\leqslant\max_{i\geqslant\log_\theta a_L}\max_{1\leqslant n\leqslant M_i}\max_{1\leqslant k\leqslant\theta^{i+1}}|U_{n+k}-U_n|/\sigma^*_{ni}\{2\log((n\vee\theta^{i+1})(\log\theta^i)/\theta^{i+1})\}^{1/2},$$

where $\sigma^{*2}_{ni}=\sum_{j=n+1}^{n+[\theta^i]}\sigma_j^2$. So we need only to show that

$$\overline{\lim_{i\to\infty}}\max_{1\leqslant n\leqslant M_i}\max_{1\leqslant k\leqslant\theta^{i+1}}|U_{n+k}-U_n|/\sigma^*_{ni}\{2\log((n\vee\theta^{i+1})(\log\theta^i)/\theta^{i+1})\}^{1/2}$$

$$\leqslant 1+\varepsilon\quad\text{a.s.}\tag{2.3.20}$$

Let $r=r(\varepsilon)$ be large enough, $R=[\theta^{i+1}/2^r]$, $n_r=[n/R]R$. Then

$$|U_{n+k}-U_n|\leqslant|U_{(n+k)_r}-U_{n_r}|+|U_{n+k}-U_{(n+k)_r}|+|U_n-U_{n_r}|.\tag{2.3.21}$$

Put $\sigma^{*2}_{ni}(r)=\sum_{j=n+1}^{n+[\theta^{i+1}]+R}\sigma_j^2$. From (2.3.17) we have

$$1\leqslant\sigma^*_{n_r,i}(r)/\sigma^*_{n,i}\leqslant 1+\varepsilon/10$$

for large $i$, provided that $r=r(\varepsilon)$ is large enough and $\theta=\theta(\varepsilon)$ is close to 1 enough. Using Kolmogorov's exponential inequality, we find

$$P\{\max_{1\leqslant n\leqslant M_i}\max_{1\leqslant k\leqslant\theta^{i+1}}|U_{(n+k)_r}-U_{n_r}|/\sigma^*_{ni}\{2\log(n(\log\theta^i)/\theta^{i+1})\}^{1/2}\geqslant 1+\varepsilon/3\}$$

$$\leqslant P\{\max_{0\leqslant j\leqslant M_i/R}\max_{1\leqslant k\leqslant\theta^{i+1}+R}|U_{jR+k}-U_{jR}|/\sigma^*_{jR,i}(r)(2\log\frac{(jR+1)\log\theta^i}{\theta^{i+1}})^{1/2}$$

$$\geqslant 1+\frac{\varepsilon}{10}\}$$

$$\leqslant\sum_{j=0}^{[M_i/R]}2\exp\{-(1+\varepsilon/10)\log\frac{(jR+1)\log\theta^i}{\theta^{i+1}}\}$$

$$\leqslant ci^{-(1+\varepsilon/10)},$$

which implies that

$$\overline{\lim_{i\to\infty}}\max_{1\leqslant n\leqslant M_i}\max_{1\leqslant k\leqslant\theta^{i+1}}|U_{(n+k)_r}-U_{n_r}|/\sigma^*_{ni}\{2\log(n(\log\theta^i)/\theta^{i+1})\}^{1/2}\leqslant 1+\varepsilon/3$$

a.s.

Similarly, for the second term of the right-hand side of (2.3.21), since

$$\sigma_{ni}^{*2}/\max_{1\leqslant k\leqslant \theta^{i+1}}\sum_{l=(n+k)_r+1}^{(n+k)_r+R}\sigma_l^2\geqslant\sigma^2[\theta^i]/(\sigma'^2\theta^{i+1}/2^r)\geqslant 3b/\varepsilon^2$$

by (2.3.17), provided that $r$ is large enough, we have

$$P\{\max_{1\leqslant n\leqslant M_i}\max_{1\leqslant k\leqslant \theta^{i+1}}|U_{n+k}-U_{(n+k)_r}|/\sigma_{ni}^*(2\log((n\vee\theta^{i+1})(\log\theta^i)/\theta^{i+1}))^{1/2}\geqslant\varepsilon/3\}$$

$$\leqslant\sum_{j=0}^{[(M_i+\theta^{i+1})/R]}P\{\max_{jR<l\leqslant(j+1)R}|U_l-U_{jR}|/(2(\sum_{l=jR+1}^{(j+1)R}\sigma_l^2)$$

$$\cdot\log\frac{((jR-\theta^{i+1})\vee\theta^{i+1})\log\theta^i}{\theta^{i+1}})^{1/2}\geqslant 2\}$$

$$\leqslant\sum_{j=0}^{2^{r+1}}2\exp\{-3\log\log\theta^i\}+\sum_{j=2^{r+1}+1}^{\infty}2\exp\{-2\log((j/2^r-1)\log\theta^i)\}$$

$$\leqslant i^{-3}2^{r+2}\log\theta+2\log\theta\sum_{j=2^{r+1}+1}^{\infty}(i(j/2^r-1))^{-2}$$

$$\leqslant i^{-2}C(r)\log\theta$$

for some $C(r)>0$, when $i$ is large enough. Hence we obtain

$$\overline{\lim_{i\to\infty}}\max_{1\leqslant n\leqslant M_i}\max_{1\leqslant k\leqslant\theta^{i+1}}|U_{n+k}-U_{(n+k)_r}|/\sigma_{ni}^*\{2\log(n(\log\theta^i)/\theta^{i+1})\}^{1/2}\leqslant\varepsilon/3\qquad\text{a.s.}$$

We have the same conclusion for the third term of the right-hand side of (2.3.21). Combining these results yields (2.3.20). (2.3.15) is proved. And (2.3.9) follows from (2.3.15) and (2.3.16). The proof of (2.3.10) is completely similar.

Next, we prove (2.3.11). It suffices by (2.3.14) and (2.3.16) to show that

$$\overline{\lim_{N\to\infty}}\beta_{NN}(U_{N+a_N}-U_N)\geqslant 1-\varepsilon\qquad\text{a.s.}\tag{2.3.22}$$

Define $N_1=1$, $N_k=N_{k-1}+a_{N_{k-1}}$ for $k\geqslant 2$. By the independence, (2.3.22) will be followed if we have

$$\sum_{k=1}^{\infty}P\{\beta_{N_kN_k}(U_{N_k+a_{N_k}}-U_{N_k})\geqslant 1-\varepsilon\}=\infty.\tag{2.3.23}$$

In fact

$$\begin{aligned}
&\sum_{k=1}^{\infty} P\{\beta_{N_k N_k}(U_{N_k+a_{N_k}} - U_{N_k}) \geqslant 1-\varepsilon\} \\
&\geqslant \sum_{k=1}^{\infty} \exp\{-(1-\varepsilon)\log(N_k \log N_k)/a_{N_k})\} \\
&\geqslant \sum_{k=1}^{\infty} (N_{k+1}-N_k)/N_k \log N_k \\
&\geqslant \sum_{k=1}^{\infty} \int_{N_k}^{N_{k+1}} \frac{1}{x\log x}\, dx = \infty .
\end{aligned}$$

i.e., (2.3.23) is true. (2.3.11) is proved. This completes the proof of Theorem 2.3.1.

Clearly, this theorem generalizes and improves Theorem 2.1.1 to a great extent. The result corresponding to Theorem 2.1.2 is the following

**Theorem 2.3.2** *Suppose that* $\{X_n\}$ *satisfies the Conditions* (i) *and* (ii′) *there exists a non-decreasing continuous function* $H(x)$, $x \geqslant 0$, *satisfying*

$$\sum_{n=1}^{\infty} P\{H(|X_n|) > bn\} < \infty \quad \textit{for some } b>0; \tag{2.3.24}$$

$$x/\log H(x) \textit{ is non-decreasing}; \tag{2.3.25}$$

$$E(H(|X_n|))^{\beta} \leqslant M < \infty \textit{ for some } \beta > 0. \tag{2.3.26}$$

*And suppose that* $\{a_n\}$ *satisfies the condition*

(a′) *there exists a sequence* $b_n \uparrow \infty$ *such that*

$$b_n(\mathrm{inv}H(n))^2 / \log n \leqslant a_n \leqslant n.$$

*Then the statments* (2.3.3)—(2.3.6) *of Theorem* 2.3.1 *are true. If we also assume that Condition* (b) *of Theorem* 2.3.1 *is satisfied, then the statements* (2.3.7) *and* (2.3.8) *are also true.*

The proof of Theorem 2.3.2 is similar to that of Theorem 2.3.1 excetp that we use the following Lemma instead of Lemma 2.3.1.

**Lemma 2.3.2** *Let $X$ be a random variable with $EX=0$. Let $a>0$, $0<\alpha\leqslant 1$. Suppose that $x/\log H(x)\ (x>0)$ is non-decreasing. Then for $0\leqslant ta\leqslant(\alpha^2/10)\log H(a)$, we have*

$$E\exp\{tXI(X\leqslant a)\}$$

$$\leqslant \exp\Big\{\frac{t^2}{2}EX^2+t^{2+\alpha/2}(E|X|^{2+\alpha})^{(4+\alpha)/(4+2\alpha)}(E(H(|X|))^{\alpha})^{\alpha/(4+4\alpha)}.$$

The proof of this lemma is similar to that of Lemma 2.3.1 and is omitted here.

The inferior limit for increments of partial sums of random variables were discussed by Lin (1990a). He obtained the following theorems.

**Theorem 2.3.3** (Lin 1990a) *Let $\{X_n, n\geqslant 1\}$ be a sequence of independent random variables. Suppose that the following conditions are satisfied:*

(i) $\varliminf_{n\to\infty}\inf_{m>0} E(X_{m+1}+\cdots+X_{m+n})^2/n>0$,

(ii) *there exist $t_0>0$ and $b>0$ such that for every $k\geqslant 1$ and $|t|\leqslant t_0$*

$$Ee^{tX_k}\leqslant b.$$

*And suppose that $\{a_n; n\geqslant 1\}$ satisfies the following conditions:*

(a) $a_n\leqslant n$, $a_n/\log n\to\infty$ as $n\to\infty$,

(b) $n/a_n$ *is non-decreasing*,

(c) $(\log n/a_n)/\log\log\log n\to\infty$ as $n\to\infty$.

*Then*

$$\varliminf_{N\to\infty}\max_{0\leqslant n\leqslant N-a_N}\gamma_{nN}|S_{n+a_N}-S_n|=1 \quad \text{a.s.} \tag{2.3.27}$$

$$\varliminf_{N\to\infty}\max_{0\leqslant n\leqslant N-a_N}\max_{1\leqslant k\leqslant a_N}\gamma_{nN}|S_{n+k}-S_n|=1 \quad \text{a.s.} \tag{2.3.28}$$

*where*

$$\gamma_{nN}=\{2\sigma_{nN}^2\log(N/\sigma_{nN}^2\log\log N)\}^{-1/2}.$$

*Proof* By Conditions (i) and (ii), it is easy to see that there exist $0<c_1\leqslant c_2<\infty$ such that

$$c_1a_N\leqslant\sigma_{nN}^2\leqslant c_2a_N \quad \text{for large } N. \tag{2.3.29}$$

First, we prove

$$\varliminf_{N\to\infty}\max_{0\leqslant n\leqslant N-a_N}\gamma_{nN}|S_{n+a_N}-S_n|\geqslant 1 \quad \text{a.s.} \tag{2.3.30}$$

Denote

$$Y_n = X_n I(|X_n| < A(\log \frac{n\log n}{a_n})/(\log \frac{n}{a_n \log\log n})),$$

where $A$ is a positive constant. For any $\varepsilon > 0$, $A = A(\varepsilon)$ shall be taken to be large enough, if

$$\varlimsup_{n \to \infty} (\log \frac{n\log n}{a_n})/(\log \frac{n}{a_n \log\log n}) < \infty .$$

Put

$$Z_n = X_n - Y_n, \quad Y_n' = Y_n - EY_n, \quad Z_n' = Z_n - EZ_n,$$

$$U_n = \sum_{k=1}^{n} Y_k', \quad V_n = \sum_{k=1}^{n} Z_k', \quad \lambda_k^2 = \mathrm{Var}\, Y_k,$$

$$\lambda_{nN}^2 = \lambda_{n+1}^2 + \cdots + \lambda_{n+a_N}^2, \quad \gamma_{nN}' = \{2\lambda_{nN}^2 \log(N/\lambda_{nN}^2 \log\log N)\}^{-1/2}.$$

By (2.3.29), there exist $0 < c_3 \leqslant c_4 < \infty$ such that

$$c_3 a_N \leqslant \lambda_{nN} \leqslant c_4 a_N \quad \text{for large } N. \tag{2.3.31}$$

It is clear that for any $\varepsilon > 0$

$$\varlimsup_{N \to \infty} |\gamma_{nN}/\gamma_{nN}' - 1| \leqslant \varepsilon \text{ uniformly in } n, \tag{2.3.32}$$

$(A = A(\varepsilon)$ large enough; and $\varepsilon = 0$ when

$$\lim_{n \to \infty} (\log \frac{n\log n}{a_n})/(\log \frac{n}{a_n \log\log n}) = \infty .$$

We can prove

$$\varlimsup_{N \to \infty} \max_{1 \leqslant n \leqslant N - a_N} \max_{1 \leqslant k \leqslant a_N} \gamma_{nN} |V_{n+k} - V_n| = 0 \quad \text{a.s.} \tag{2.3.33}$$

In fact, for $|t| \leqslant t_0/2$

$$Ee^{tZ_n'} \leqslant \exp\left\{ \frac{t^2}{24} \varepsilon^2 c_3 \frac{\log(n/a_n \log\log n)}{\log((n\log n)/a_n)} \right\}.$$

Noting that under Condition (a), for large $N$

$$\frac{a_N \log(N/a_N \log\log N)}{\log^2((N\log N)/a_N)}$$

$$\geqslant \begin{cases} \dfrac{a_N}{\log N} \dfrac{\log(N/(\log^2 N)\log\log N)}{\log N}, & \text{if } a_N < \log^2 N \\ \log(N/a_N \log\log N), & \text{if } a_N \geqslant \log^2 \mathrm{N}. \end{cases}$$

Hence, by Conditions (a) and (c)

$$\frac{a_N \log (N/a_N \log \log N)}{\log^2 ((N\log N)/a_N)} \to \infty, \ N \to \infty. \tag{2.3.34}$$

Use Lemma 2.2.1, (2.3.33) holds true. From (2.3.32), (2.3.30) is equivalent to

$$\varliminf_{N \to \infty} \max_{0 \leqslant n \leqslant N-a_N} \gamma'_{nN} | U_{n+a_N} - U_n | \geqslant 1 \quad \text{a.s.} \tag{2.3.35}$$

Write

$$\begin{aligned} &P\{ \max_{0 \leqslant n \leqslant N-a_N} \gamma'_{nN} | U_{n+a_N} - U_n | < 1-\varepsilon \} \qquad (2.3.36) \\ &\leqslant \prod_{j=0}^{[N/a_N]-1} \{ 1 - P\{ \gamma'_{ja_N, N} | U_{(j+1)a_N} - U_{ja_N} | \geqslant 1-\varepsilon \}\}. \end{aligned}$$

By (2.3.34), using Lemma 2.2.1 for $\{ Y'_n \}$, we get

$$\begin{aligned} &P\{ \gamma'_{ja_N, N} | U_{(j+1)a_N} - U_{ja_N} | \geqslant 1-\varepsilon \} \qquad (2.3.37) \\ &\geqslant \exp\{ -(1-\varepsilon)(1-\varepsilon)^2 \log (N/\lambda^2_{ja_N, N} \log \log N) \} \\ &\geqslant (\lambda^2_{ja_N, N} (\log \log N)/N)^{1-\varepsilon} \geqslant c \left( \frac{a_N \log \log N}{N} \right)^{1-\varepsilon}. \end{aligned}$$

Then, using Condition (c)

$$\begin{aligned} &P\{ \max_{0 \leqslant n \leqslant N-a_N} \gamma'_{nN} | U_{n+a_N} - U_n | < 1-\varepsilon \} \qquad (2.3.38) \\ &\leqslant \prod_{j=0}^{[N/a_N]-1} \left(1 - c \left( \frac{a_N \log \log N}{N} \right)^{1-\varepsilon} \right) \\ &\leqslant \exp\left\{ -c \left( \frac{N}{a_N} \right)^{\varepsilon} (\log \log N)^{1-\varepsilon} \right\} \leqslant \log^{-3} N. \end{aligned}$$

Let $N = 2^{\sqrt{k}}$, it follows that

$$\varliminf_{k \to \infty} \max_{0 \leqslant n \leqslant N_k - a_{N_k}} \gamma'_{nN_k} | U_{n+a_{N_k}} - U_n | \geqslant 1 \quad \text{a.s.} \tag{2.3.39}$$

For any given $N$, there exists an integer $k$ such that $N_k \leqslant N \leqslant N_{k-1}$. Write

$$\begin{aligned} &\max_{0 \leqslant n \leqslant N-a_N} \gamma'_{nN} | U_{n+a_N} - U_n | \\ &\geqslant \max_{0 \leqslant n \leqslant N_k - a_{N_k}} \gamma'_{nN_k} | U_{n+a_{N_k}} - U_n | - \max_{1 \leqslant i \leqslant N_k} \max_{1 \leqslant j \leqslant a_{N_{k+1}} - a_{N_k}} \gamma'_{nN_{k+1}} | U_{i+j} - U_i | \\ &=: I_1(k) - I_2(k). \qquad (2.3.40) \end{aligned}$$

By Conditions (i) and (ii), for $n$ uniformly

$$\lambda_{nN_k}/\lambda_{nN_{k+1}} \to 1 \quad k \to \infty ,$$

then

$$\varliminf_{k \to \infty} I_1(k) \geqslant 1 \qquad \text{a.s.} \tag{2.3.41}$$

In order to estimate $I_2(k)$, denote

$$M_k = N_k + a_{N_{k+1}} - a_{N_k},\ \lambda_{ik}'^2 = \lambda_{i+1} + \cdots + \lambda_{i+a_{N_{k+1}}-a_{N_k}}.$$

By Theorem 2.3.1, it is easy to see that

$$\varlimsup_{k \to \infty} \max_{1 \leqslant i \leqslant N_k} \max_{1 \leqslant j \leqslant a_{N_{k+1}}-a_{N_k}} \frac{|U_{i+j} - U_i|}{\lambda_{ik}'\{2(\log(M_k/\lambda_{ik}'^2) + \log\log M_k)\}^{1/2}} \leqslant 1 \qquad \text{a.s.} \tag{2.3.42}$$

By Conditions (a)—(c), it is easy to check that

$$\begin{aligned}
&\gamma'_{nN_{k+1}}\{\lambda_{ik}'^2(\log(M_k/\lambda_{ik}'^2) + \log\log M_k)\} \\
&\leqslant c\,\frac{a_{N_{k+1}} - a_{N_k}}{a_{N_{k+1}}}\ \frac{\log(N_k/(a_{N_{k+1}} - a_{N_k})) + \log\log N_k}{\log(N_{k+1}/a_{N_{k+1}}\log\log N_{k+1})} \\
&\leqslant c\Big(1 - \frac{N_k}{N_{k+1}}\Big)(\log N_k)\Big(\log \frac{N_{k+1}}{a_{N_{k+1}}\log\log N_{k+1}}\Big)^{-1} \to 0,
\end{aligned}$$

uniformly in $n$ and $i$ as $k \to \infty$. Hence

$$\varlimsup_{k \to \infty} I_2(k) = 0 \qquad \text{a.s.} \tag{2.3.43}$$

Combining it with (2.3.41) and (2.3.42), we obtain (2.3.30).

Next, we prove

$$\varlimsup_{N \to \infty} \max_{0 \leqslant n \leqslant N-a_N} \max_{1 \leqslant k \leqslant a_N} \gamma_{nN}|S_{n+k} - S_n| \leqslant 1 \qquad \text{a.s.} \tag{2.3.44}$$

Let $r = r(\varepsilon)$ be a positive constant specified later on. Put

$$c_N = [\,a_N/r\,],\ n_i = ic_N. \tag{2.3.45}$$

Write

$$\begin{aligned}
&\max_{0 \leqslant n \leqslant N-a_N} \max_{1 \leqslant k \leqslant a_N} \gamma_{nN}|S_{n+k} - S_n| \\
&\leqslant \max_{1 \leqslant i \leqslant N/c_N} \max_{n_{i-1} \leqslant n < n_i} \max_{1 \leqslant k \leqslant a_N} \gamma_{nN}(|S_{n+k} - S_{n_i}| + |S_n - S_{n_i}|) \\
&\leqslant 2 \max_{1 \leqslant i \leqslant N/c_N} \max_{n_{i-1} \leqslant n < n_i} \gamma_{nN}|S_n - S_{n_i}|
\end{aligned} \tag{2.3.46}$$

$$+ \max_{1 \leqslant i \leqslant N/C_N} \max_{n_{i-1} \leqslant n < n_i} \max_{1 \leqslant k \leqslant a_N} \gamma_{nN} | S_{n_{i+k}} - S_{n_i} |.$$

Thus, we have

$$\begin{aligned}
& P\{ \max_{0 \leqslant n \leqslant N - a_N} \max_{1 \leqslant k \leqslant a_N} \gamma_{nN} | S_{n+k} - S_n | \geqslant 1 + \varepsilon \} \\
& \leqslant P\{ \max_{1 \leqslant i \leqslant N/C_N} \max_{n_{i-1} \leqslant n < n_i} \gamma_{nN} | S_n - S_{n_i} | \geqslant \varepsilon/4 \} \\
& \quad + P\{ \max_{1 \leqslant i \leqslant N/C_N} \max_{n_{i-1} \leqslant n < n_i} \max_{1 \leqslant k \leqslant a_N} \gamma_{nN} | S_{n_i+k} - S_{n_i} | \geqslant 1 + \varepsilon/2 \} \\
& \leqslant \frac{N}{c_N} \max_{1 \leqslant i \leqslant N/C_N} P\{ \max_{n_{i-1} \leqslant n < n_i} \gamma_{nN} | S_n - S_{n_i} | \geqslant \varepsilon/4 \} \\
& \quad + \frac{N}{c_N} \max_{1 \leqslant i \leqslant N/C_N} P\{ \max_{n_{i-1} \leqslant n < n_i} \max_{1 \leqslant k \leqslant a_N} \gamma_{nN} | S_{n_i+k} - S_{n_i} | \geqslant 1 + \varepsilon/2 \} \\
& = : J_1(N) + J_2(N).
\end{aligned}$$

Estimate $J_1(N)$. From the conditions of the theorem, there exists $T > 0$ such that

$$E \exp(tX_k) \leqslant \exp(g\sigma_k^2 t^2/2)$$

with $g = 1 + \varepsilon/4$ for any $|t| \leqslant T$ and every $k \geqslant 1$. By taking

$$x = \frac{\varepsilon}{4} \max_{n_{i-1} \leqslant n \leqslant n_i} \gamma_{nN}^{-1}, \quad G = g \sum_{j=n_{i-1}}^{n_i - 1} \sigma_j^2$$

in Lemma 2.2.1, it follows from Condition (a) that $0 < x \leqslant GT$ for large $N$. Using Lemma 2.2.1, we have

$$\begin{aligned}
J_1(N) & \leqslant \frac{N}{c_N} \max_{1 \leqslant i \leqslant N/C_N} \exp\{ -\frac{\varepsilon^2}{16} ( \min_{n_{i-1} \leqslant n < n_i} \sigma_{nN}^2 \log(N/\sigma_{nN}^2 \log\log N)) \\
& \quad /(g \sum_{j=N_{i-1}}^{N_i - 1} \sigma_j^2 ) \} \\
& \leqslant \frac{N}{c_N} \exp\{ -c\varepsilon^2 r \log(N/a_N \log\log N) \} \\
& \leqslant \frac{rN}{a_N} \left( \frac{a_N \log\log N}{N} \right)^{c\varepsilon^2 r} \leqslant r \frac{a_N}{N} (\log\log N)^2
\end{aligned}$$

for large $r$. Estimate $J_2(N)$. Letting $r$ be large enough, we have

$$|( \min_{n_{i-1} \leqslant n < n_i} \sigma_{nN}^2 )/( \sum_{j=n_{i+1}}^{n_i + a_N} \sigma_j^2 ) - 1 | \leqslant \varepsilon/4$$

for large $N$. Similarly to the estimation for $J_1(N)$, we have

$$J_2(N) \leqslant \frac{N}{c_N} \max_{1 \leqslant i \leqslant N/C_N} \exp\{ -(1+\frac{\varepsilon}{2})^2 ( \min_{n_{i-1} \leqslant n < n_i} \sigma_{nN}^2 \log(N/\sigma_{nN}^2 \log\log N)) / (g \sum_{j=n_i+1}^{n_i+a_N} \sigma_j^2 ) \}$$

$$= r\left(\frac{a_N}{N}\right)^{\varepsilon/4} (\log\log N)^{1+\varepsilon/4}.$$

It follows from (2.3.47) that

$$P\{ \max_{1 \leqslant n \leqslant N-a_N} \max_{1 \leqslant k \leqslant a_N} \gamma_{nN} | S_{n+k} - S_n | \geqslant 1+\varepsilon \} \leqslant 2r\left(\frac{a_N}{N}\right)^{\varepsilon/4} (\log\log N)^2.$$

By assumption (c), there exists a subsequence $\{ N_j \}$ of positive integers such that

$$\lim_{j \to \infty} P\{ \max_{0 \leqslant n \leqslant N_j - a_{N_j}} \max_{1 \leqslant k \leqslant a_{N_j}} \gamma_{n N_j} | S_{n+k} - S_n | \geqslant 1+\varepsilon \} = 0,$$

which implies (2.3.44).

*Remark* 2.3.1 When the moment generating functions of $\{ X_n \}$ do not exist, one replaces Condition (ii) by

(ii$'$) there exists a non-decreasing continuous function $H(x)$, $x \geqslant 0$, satisfying

$$\sum_{n=1}^{\infty} P\{ H(|X_n|) > bn \} < \infty \quad \text{for any } b \geqslant 0,$$

$$E(H(|X_n|))^{\beta} \leqslant M < \infty \qquad \text{for every } n \text{ and any } \beta < 1,$$

$$H(x)/x^{2+\alpha} \text{ is increasing for some } \alpha > 0,$$

$$\sum_{n=1}^{\infty} (\operatorname{inv} H(n))^{-A} < \infty \qquad \text{for some } A > 0,$$

$$\varliminf_{x \to \infty} H(\varepsilon x)/H(x) > 0 \qquad \text{for any } \varepsilon > 0,$$

and replaces Condition (a) by

(a$'$) there exists a number $a > 0$ such that

$$a(\operatorname{inv} H(n))^2 / \log n \leqslant a_n \leqslant n.$$

That is to say, if $\{ X_n \}$ satisfies (i) and (ii$'$) and $\{ a_n \}$ satisfies (a$'$), (b) and (c), then the conclusions of Theorem 2.3.3 are still true.

## 2.4 On the Increments Without Moment Hypotheses

All of the limit results on increments of partial sums of a sequence of random variables in the above two sections require the existence of either moment generating functions or moments larger than order 2. It has been pointed out that strong limit theorems depend (in principle) on probabilities rather than moments (cf. Klass, Tomkins 1984). Some probabilisto have discussed the law of the iterated logarithm without the moment condition (cf. Klass, Teicher 1977 and Tomkins 1980). The a.s. limiting behavior of increments of partial sums can be regarded more as generalization and elegance for the law of the iterated logarithm. So it is interesting to know how big the increments of partial sums without moment hypotheses are. Lin (1990b) first considered this problem. His theorem is a generalization of the results with moment hypotheses. Lin and Shao (1990) weakened the conditions of this theorem to a great extent.

Let $\{X_n; n \geqslant 1\}$ be a sequence of independent, but not necessarily identically distributed random variables and $\{a_n; n \geqslant 1\}$ be a sequence of positive integers tending to infinity. Denote $S_n = \sum_{i=1}^{n} X_i$. Furthermore, let $\{B_{nN},\ n=0, 1, \cdots, N;\ N=1, 2, \cdots\}$ be a double sequence of positive numbers, which is non-decreasing on $N$ for fixed $n$ and tends to infinity as $N \to \infty$ uniformly in $n$. Denote $B_N = B_{ON}$, $b_N^2 = 2\{\log(B_{N+a_N}^2/B_{a_N}^2) + \log\log B_{a_N}^2\}$. For every $N$ and $n + a_n \leqslant N < n + 1 + a_{n+1}$, define $B'_N = B_{n, n+a_n}$ (so $B'_{N+a_N} = B_{N, N+a_N}$), $b_N'^2 = 2\{\log(B_{n+a_n}^2/B_N'^2) + \log\log B_N'^2\}$. And define for $\varepsilon > 0$

$$X_{j\varepsilon} = (X_j \vee (-\varepsilon B'_j b_j'^{-1})) \wedge (\varepsilon B'_j b'_j),$$

$$T_N(\varepsilon) = B_{N+a_N}^{-2} \sum_{j=1}^{N+a_N} \mathrm{Var}(X_{j\varepsilon}), \quad T_{nN}(\varepsilon) = B_{n, n+a_N}^{-2} \sum_{j=n+1}^{n+a_N} \mathrm{Var}(X_{j\varepsilon}),$$

$$T_-^2 = \lim_{\varepsilon \downarrow 0} \lim_{N \to \infty} T_N(\varepsilon) \wedge T_{NN}(\varepsilon), \quad T_+^2 = \lim_{\varepsilon \downarrow 0} \overline{\lim_{N \to \infty}} \max_{0 \leqslant n \leqslant N} T_{nN}(\varepsilon).$$

Assume $T_+ < \infty$.

**Theorem 2.4.1** *Suppose that the following conditions are satisfied: for any* $\varepsilon>0$,

(i) $\sum_{n=1}^{\infty} P\{|X_n|\geqslant \varepsilon B_n' b_n'\}<\infty$;

(ii) $\lim_{N\to\infty} \max_{0\leqslant n\leqslant N} \max_{1\leqslant k\leqslant a_N} (B_{a_N} b_N)^{-1} |\sum_{j=n+1}^{n+k} E\{X_j I(|X_j|\leqslant \varepsilon B_j' b_j')\}|=0$;

(iii) $\lim_{\varepsilon\downarrow 0} \lim_{N\to\infty} \max_{0\leqslant n\leqslant N} \sum_{j=n+1}^{n+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1}<|X_j|\leqslant \varepsilon B_j' b_j')$

$/(B_{a_N}^2((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-\beta})<\infty$ for some $\beta>0$;

(iv) $\lim_{N\to\infty} (\max_{0\leqslant n\leqslant N} B_{n,n+a_N})/(\min_{0\leqslant n\leqslant N} B_{n,n+a_N})<\infty$;

(v) $B_{N+a_N}\leqslant AB_{N-1+a_N-1}$ and $B_{a_N}\leqslant AB_{a_{N-1}}$ for some $A>0$ and every $N\geqslant 2$.

*Then*

$$T_-\leqslant \lim_{N\to\infty} \frac{|S_{N+a_N}-S_N|}{B_{N,N+a_N} b_N} \leqslant \lim_{N\to\infty} \mathit{lim}_{1\leqslant n\leqslant N} \mathit{lim}_{1\leqslant k\leqslant a_N} \frac{|S_{n+k}-S_n|}{B_{n,a+a_N} b_N}\leqslant T \quad \text{a.s.}$$

*Example.* Let $\{X_n\}$ be a sequence of i.i.d. random variables with

$P\{X=-\sqrt{n}\}=P(X=\sqrt{n})=a/n^2\log n$, $n=2,3,\cdots$,

where $a=\frac{1}{2}(\sum_{n=2}^{\infty}\frac{1}{n^2\log n})^{-1}$. It is clear that $EX_n^2=\infty$. Take $a_n=n^2$, $B_{n,n+k}^2=2ak\log\log k$. So $B_{n,n+a_N}^2=2aa_N\log\log a_N\sim 2aN^2\log\log N$. Then we have $B_{a_N}^2\sim 2aN^2\log\log N$, $b_N^2\sim 2\log\log N$, $B_N'^2\sim 2aN\log\log N$, $b_N'^2\sim 2\log\log N$ by the definitions. Thus

$$\sum_{i=n+1}^{n+a_N} \mathrm{Var}(X_{i\varepsilon})\sim \sum_{i=n+1}^{n+a_N}\Big(\sum_{k=2}^{[\varepsilon^2 ai]}\frac{2a}{k\log k}+\sum_{k=[\varepsilon^2 a_i]}^{\infty}\frac{2a^2\varepsilon^2 i}{k^2\log k}\Big)\sim 2aa_N\log\log a_N.$$

So $T_-=T_+=1$. It is not difficult to show that Conditions (i)—(v) are satisfied. We verify only (iii).

$$\sum_{j=n+1}^{n+k} EX_j^2(\varepsilon B_j' b_j'^{-1}<|X_j|\leqslant \varepsilon B_j' b_j')$$

$$\leqslant \sum_{j=n+1}^{n+k}\ \sum_{i=[\varepsilon^2 aj]}^{[4\theta^2 aj(\log\log j)^2]} \frac{2a}{i\log i}$$

$$\leqslant 3a \sum_{j=n+1}^{n+k} \log \left(1 + \frac{2\log (2\log \log j)}{\log (\varepsilon^2 aj)}\right)$$

$$\leqslant 7ak \frac{\log \log \log k}{\log k}$$

for $0 \leqslant n \leqslant N, k = a_N$, which implies (iii).

In order to prove Theorem 2.4.1, we need the following Lemma.

**Lemma 2.4.1** *Let $\{\xi_n; n \geqslant 1\}$ be a sequence of independent random variables with $E\xi_n = 0$. And let $\{a_n; n \geqslant 1\}$ be a sequence of positive integers tending to infinity. Suppose that there exists a double sequence $\{\sigma_{nN}, n = 0, 1, \cdots, N; N = 1, 2, \cdots\}$ of positive numbers, which is non-decreasing on $N$ for fixed $n$ and tends to infinity as $N \to \infty$ uniformly in $n$. Put* $\sigma_N^2 = \min_{0 \leqslant n \leqslant N} \sigma_{naN}^2, \beta_{nN} = \{2\sigma_{na_N}^2 \log (\sigma_{0,N+a_N}^2 / \sigma_{na_N}^2) + \log \log \sigma_{na_N}^2\}^{1/2}$, If

(a) $\sigma_{0a_n} \leqslant A\sigma_{0a_{n-1}}$ *for some $A > 0$;*

(b) $\overline{\lim}_{N \to \infty} \max_{1 \leqslant n \leqslant N} \sum_{i=n+1}^{n+a_N} E\xi_i^2 / \sigma_{na_N}^2 \leqslant 1$;

(c) *there exists a $\varepsilon > 0$ such that*

$$|\xi_N| \leqslant \varepsilon \{\sigma_N^2 / (\log (\sigma_{0,N+a_N}^2 / \sigma_N^2) + \log \log \sigma_N^2)\}^{1/2},$$

*then, there is a $c(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that*

$$\overline{\lim}_{N \to \infty} \max_{0 \leqslant n \leqslant N} \max_{1 \leqslant k \leqslant a_N} \left| \sum_{i=n+1}^{n+k} \xi_i \right| / \beta_{nN} \leqslant 1 + c(\varepsilon) \quad \text{a.s.}$$

Proof of Theorem 2.4.1.

Without loss of generality, we assume $T_- > 0$.

At first, we give some facts applied in the sequel.

Take such an $n$ for $N$ that $a_{n-1} < N + a_N \leqslant a_n$. By Condition (v),

$$\sum_{j=1}^{N+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \leqslant \varepsilon B_j' b_j') / B_{N+a_N}^2$$

$$\leqslant \sum_{j=1}^{a_n} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \leqslant \varepsilon B_j' b_j') / B_{a_n-1}^2$$

$$\leqslant A^2 \sum_{j=1}^{a_n} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \leqslant \varepsilon B_j' b_j') / B_{a_n}^2$$

and similarly

$$\sum_{j=1}^{N+a_N} \operatorname{Var}(X_{j\varepsilon})/B_{N+a_N}^2 \leqslant A^2 \sum_{j=1}^{a_n} \operatorname{Var}(X_{j\varepsilon})/B_{a_n}^2 .$$

Hence

$$\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}} \sum_{j=1}^{N+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \leqslant \varepsilon B_j' b_j')/B_{N+a_N}^2 \tag{2.4.1}$$

$$\leqslant A^2 \overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}} \max_{0\leqslant n\leqslant N} \Big\{ \sum_{j=n+1}^{n+a_N} EX_j^2 I(\varepsilon B_j' b_j'^{-1} < |X_j| \leqslant \varepsilon B_j' b_j')$$

$$\bigwedge \big[B_{a_N}^2 ((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-\beta}\big]\Big\}((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-\beta}$$

$$=0$$

by Condition (iii), and

$$\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}} \sum_{j=1}^{N+a_N} \operatorname{Var}(X_{j\varepsilon})/B_{N+a_N}^2 \tag{2.4.2}$$

$$\leqslant A^2 \overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}} \max_{0\leqslant n\leqslant N} T_{nN}(\varepsilon) = A^2 T^2.$$

Define $N_j$ by $N_j = \min\{n : B_{a_n} b_n \geqslant 2^j\}$, which implies that

$$B_{a_{N_j-1}} b_{N_j-1} < 2^j \leqslant B_{a_{N_j}} b_{N_j} .$$

From Condition (v), we can find a constant $C>0$ such that for every $N\geqslant 2$

$$b_N \leqslant C\, b_{N-1} .$$

Therefore

$$B_{a_{N_j}} b_{N_j} \leqslant AC\, B_{a_{N_j-1}} b_{N_j-1}$$

and

$$2^j \leqslant B_{a_{N_j}} b_{N_j} \leqslant AC\, 2^j .$$

The latter implies

$$B_{a_{N_{j+1}}} b_{N_{j+1}} / B_{a_{N_j}} b_{N_j} \leqslant 2AC. \tag{2.4.3}$$

Furthermore, we have either

$$B_{a_{N_j}} \geqslant 2^{j/2} \tag{2.4.4}$$

or

$$B_{a_{N_j}} < 2^{j/2},\ b_{N_j} \geqslant 2^{j/2}. \tag{2.4.5}$$

If (2.4.4) is true, then

$$(B_{a_{N_j}}/B_{N_j+a_{N_j}})^{\varepsilon}/\log^{1+\varepsilon}B_{a_{N_j}}$$

$$\leqslant(\log B_{a_{N_j}})^{-(1+\varepsilon)}\leqslant(\frac{1}{2}j\log 2)^{-(1+\varepsilon)};$$

if (2.4.5) is true, then

$$\log(B^2_{N_j+a_{N_j}}/B^2_{a_{N_j}})\geqslant 2^{j-1}-\log\log 2^j\geqslant 2^{j-2}$$

which implies that

$$(B_{a_{N_j}}/B_{N_j+a_{N_j}})^{\varepsilon}/\log^{1+\varepsilon}B_{a_{N_j}}\leqslant j^{-(1+\varepsilon)}.$$

In any way, we have

$$\sum_{j=1}^{\infty}(B_{a_{N_j}}/B_{N_j+a_{N_j}})^{\varepsilon}/\log^{1+\varepsilon}B_{a_{N_j}}<\infty. \tag{2.4.6}$$

Moreover, there exist $\delta>0$ and $Q>0$ such that

$$\overline{\lim_{N\to\infty}}\sum_{j=1}^{N+a_N}\operatorname{Var}(X_{j\varepsilon})/\sum_{j=1}^{N-1+a_{N-1}}\operatorname{Var}(X_{j\varepsilon})\leqslant Q \tag{2.4.7}$$

uniformly in $0<\varepsilon<\delta$ since by Condition (v) and (2.4.2)

$$\sum_{j=1}^{N+a_N}\operatorname{Var}(X_{j\varepsilon})/\sum_{j=1}^{N-1+a_{N-1}}\operatorname{Var}(X_{j\varepsilon})$$

$$=(B^2_{N+a_N}/B^2_{N-1+a_{N-1}})(\sum_{j=1}^{N+a_N}\operatorname{Var}(X_{j\varepsilon})/B^2_{N+a_N})$$

$$/(\sum_{j=1}^{N-1+a_{N-1}}\operatorname{Var}(X_{j\varepsilon})/B^2_{N-1+a_{N-1}})$$

$$\leqslant A^2\cdot 2A^2T_+^2/(T_-^2/2)=4A^4T_+^2/T_-^2$$

provided that $N$ is large enough.

Using these facts, we proceed to prove the conclusion of the theorem.

For given $\delta>0$, let $\varepsilon=\varepsilon(\delta)$ be indicated later. Define $c_n=\varepsilon B_n' b_n'^{-1}$, $d_n=\varepsilon B_n' b_n'$ and $X_n'=X_{n\varepsilon}$, $Y_n=(X_n-c_n\operatorname{sign}X_n)I(c_n<|X_n|\leqslant d_n)$, $Z_n=X_n'+Y_n$, $S_n'=\sum_{k=1}^{n}(X_k'-EX_k')$, $U_n=\sum_{k=1}^{n}(Y_k-EY_k)$, $V_n=\sum_{k=1}^{n}(Z_k-EZ_k)$. Then

$$Z_n=X_n I(|X_n|\leqslant d_n)+c_n\operatorname{sign}X_n I(|X_n|>d_n),$$

$$|X_n-Z_n|\leqslant|X_n|I(|X_n|>d_n).$$

So, as a result of condition (i),

$$P\{ X_n \neq Z_n ,\mathrm{i,o.} \} =0 . \tag{2.4.8}$$

Hence we may only consider $Z_n$ instead of $X_n$. From Conditions (i) and (ii) and the definition of $c_n$, we have

$$\lim_{N\to\infty} \max_{1\leqslant n\leqslant N} \max_{1\leqslant k\leqslant a_N} \frac{1}{B_{n,n+a_N} b_N} \left| \sum_{j=n+1}^{n+k} EZ_j \right| =0. \tag{2.4.9}$$

Combining (2.4.9) with (2.4.8) implies that the conclusion of the theorem is equivalent to

$$T_- \leqslant \overline{\lim_{N\to\infty}} \frac{| V_{N+a_N} - V_N |}{B_{N,N+a_N} b_N} \leqslant \overline{\lim_{N\to\infty}} \max_{1\leqslant n\leqslant N} \max_{1\leqslant k\leqslant a_N} \frac{| V_{n+k} - V_n |}{B_{n,n+a_N} b_N} \leqslant T_+ \quad \text{a.s.} \tag{2.4.10}$$

As a first step, we prove

$$\overline{\lim_{N\to\infty}} \max_{1\leqslant n\leqslant N} \max_{1\leqslant k\leqslant a_N} \frac{| U_{n+k} - U_n |}{B_{n,n+a_N} b_N} \leqslant \delta \quad \text{a.s.} \tag{2.4.11}$$

Let $r=r(\delta)$ be a positive integer indicated later on. Put $D_N = B^2_{a_N}((B^2_{N+a_N}/B^2_{a_N})\log B^2_{a_N})^{-\beta}$ and $Y_0=0$. Define a function of $n$, $N$ and $r$ as follows: $n_r = \max\{ k : \sum_{j=1}^{k} \mathrm{Var}(Y_j) \leqslant iD_N/r$, where $i$ satisfies $\frac{i}{r} D_N \leqslant \sum_{j=1}^{n} \mathrm{Var}\, Y_j \leqslant \frac{i+1}{r} D_N \}$. Put $U_0=0$. Write

$$| U_{n+k} - U_n | \leqslant | U_{n+k} - U_{(n+k)_r} | + | U_{(n+k)_r} - U_{n_r} | + | U_n - U_{n_r} |. \tag{2.4.12}$$

Consider the first term of the right-hand side. By Condition (iv), there exists a constant $H>0$ such that

$$(\max_{0\leqslant n\leqslant N} B_{n,n+a_N})/(\min_{0\leqslant n\leqslant N} B_{n,n+a_N}) \leqslant H \tag{2.4.13}$$

for every $N$. Then we have

$$P\left\{ \max_{1\leqslant n\leqslant N} \max_{1\leqslant k\leqslant a_N} \frac{1}{B_{n,n+a_N} b_N} | U_{n+k} - U_{(n+k)_r} | \geqslant \frac{\delta}{2ACH} \right\} \tag{2.4.14}$$

$$\leqslant \left\{ r \sum_{j=1}^{N+a_N} \mathrm{Var} Y_j / D_N + 1 \right\} \max_{1\leqslant n\leqslant N+a_N} P\left\{ | U_n - U_{n_r} | \geqslant \frac{\delta}{2ACH^2} B_{a_N} b_N \right\}.$$

From Condition (iii)

$$\sum_{j=1}^{N+a_N} \operatorname{Var} Y_j$$

$$\leqslant 2 \sum_{j=1}^{N+a_N} \{ EX_j^2 I(c_j < |X_j| \leqslant d_j) + c_j^2 P(c_j < |X_j| \leqslant d_j)\} \qquad (2.4.15)$$

$$\leqslant 2M\, B_{N+a_N}^2$$

for some $M > 0$. Estimate the probability in the right-hand side of (2.4.14) Let

$$\varepsilon = \beta\delta/(16ACH^3), \quad t = \beta b_N/(4\varepsilon H) = 4ACH^2 b_N/\delta .$$

Then, for $j \leqslant N + a_N$, we have $d_j \leqslant \varepsilon H B_{a_N} b_N$ and

$$E \exp\{ t(Y_j - EY_j)/B_{a_N} \}$$

$$\leqslant 1 + \frac{t^2}{2B_{a_N}^2} \operatorname{Var} Y_j \left\{ 1 + \frac{1}{3}\left(\frac{2td_j}{B_{a_N}}\right) + \frac{1}{12}\left(\frac{2td_j}{B_{a_N}}\right)^2 + \cdots \right\}$$

$$\leqslant 1 + \frac{t^2}{2B_{a_N}^2} \operatorname{Var} Y_j \cdot \exp\left\{ \frac{\beta}{4} b_N^2 \right\}$$

$$\leqslant \exp\left\{ \frac{8A^2C^2H^4}{\delta^2 B_{a_N}^2} b_N^2 \left( \frac{B_{N+a_N}^2}{B_{a_N}^2} \log B_{a_N}^2 \right)^{\beta/2} \operatorname{Var} Y_j \right\}.$$

Put $l_n = \max\{ m : m_r = n_r \}$. Using Lévy's maximum inequality, we obtain

$$P\{ \max_{m_r = n_r} |U_m - U_{n_r}| \geqslant \frac{\delta}{2ACH^2} B_{a_N} b_N \} \qquad (2.4.16)$$

$$\leqslant 2P\{ |U_{l_n} - U_{n_r}| \geqslant \frac{\delta}{2ACH^2} B_{a_N} b_N - 2\sqrt{\operatorname{Var}(U_{l_n} - U_{n_r})} \}$$

$$\leqslant 2P\{ |U_{l_n} - U_{n_r}| \geqslant \frac{\delta}{3ACH^2} B_{a_N} b_N \}$$

$$\leqslant 2\exp\left( -\frac{\delta t b_N}{3ACH^2} \prod_{j=n_r+1}^{l_n} E\exp\left\{ \frac{t}{B_{a_N}} (Y_j - EY_j) \right\} \right.$$

$$\leqslant 2\exp(-4b_N^2/3 + o(b_N^2))\}$$

$$\leqslant ((B_{N+a_N}^2/B_{a_N}^2)\log B_{a_N}^2)^{-2}$$

for every large $N$, where the definition of $n_r$ and Condition (iv) are used for the last but two inequalities. Combining (2.4.14), (2.4.15) and (2.4.16) yields that

$$P\{\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\frac{1}{B_{n,n+a_N}b_N}|U_{n+k}-U_{(n+k)_r}|\geqslant\delta/2ACH\}\tag{2.4.17}$$

$$\leqslant 5rM(B^2_{a_N}/B^2_{N+a_N})^{1-\beta}/(\log B^2_{a_N})^{2-\beta}.$$

By (2.4.6), we have

$$\sum_{j=1}^{\infty}P\{\max_{1\leqslant n\leqslant N_j}\max_{1\leqslant k\leqslant a_{N_j}}\frac{1}{B_{n,n+a_{N_j}}b_{N_j}}|U_{n+k}-U_{(n+k)_r}|\geqslant\delta/2ACH\}<\infty$$

which implies that

$$\overline{\lim_{j\to\infty}}\max_{1\leqslant n\leqslant N_j}\max_{1\leqslant k\leqslant a_{N_j}}\frac{1}{B_{n,n+a_{N_j}}b_{N_j}}|U_{n+k}-U_{(n+k)_r}|\leqslant\delta/2ACH\quad\text{a.s.}\tag{2.4.18}$$

Furthermore, by noting that the ranges in two max's in (2.4.18) enlarge as $j$ increases and by using (2.4.3) and (2.4.13), (2.4.18) implies that

$$\overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\frac{1}{B_{n,n+a_N}b_N}|U_{n+k}-U_{(n+k)_r}|\leqslant\delta\quad\text{a.s.}\tag{2.4.19}$$

The second and the third terms of the right-hand side of (2.4.12) can be treated by the same procedure except that Condition (iii) is applied for the second term, and we have the similar inequalities. (2.4.11) is proved. Thus inequality (2.4.10) is equivalent to

$$(1-2\delta)T_-\leqslant\overline{\lim_{N\to\infty}}\frac{|S'_{N+a_N}-S'_N|}{B_{N,N+a_N}b_N}$$

$$\leqslant\overline{\lim_{N\to\infty}}\max_{1\leqslant n\leqslant N}\max_{1\leqslant k\leqslant a_N}\frac{|S'_{n+k}-S'_n|}{B_{n,n+a_N}b_N}\leqslant(1+3\delta)T\quad\text{a.s.}\tag{2.4.20}$$

for any $0<\delta<1/2$.

It is easy to verify that the condition of Lemma 2.4.1 is satisfied for $\{X'_n\}$. Consequently we have the right inequality of (2.4.20) from the definition of $T_+$.

Next, we prove the left of (2.4.20). Let $v^2_{n,n+a_N}=\sum\limits_{j=n+1}^{n+a_N}\operatorname{Var}X'_j$, $v^2_{a_N}=$

$\max_{1\leqslant n\leqslant N} v^2_{n,n+a_N}$. For $j\leqslant N+a_N$,

$$|X_j'-EX_j'|\leqslant 2c_j\leqslant 2c_{N+a_N}=2\varepsilon b'^{-1}_{N+a_N}(B_{N,N+a_N}/v_{N,N+a_N})v_{N,N+a_N}. \quad (2.4.21)$$

Noting $0<T_-<\infty$, we have $B_{N,N+a_N}b_N/v_{N,N+a_N}\to\infty$ as $N\to\infty$ and $(\varepsilon b'^{-1}_{N+a_N}B_{N,N+a_N}v^{-1}_{N,N+a_N})(B_{N,N+a_N}b_N v^{-1}_{N,N+a_N})\leqslant Q\varepsilon$ for some constant $Q>0$ and every $N\geqslant 1$. Hence, if let $\varepsilon=\varepsilon(\delta)$ be small enough, we can use Lemma 2.2.1 (b) and obtain that

$$P\left\{\frac{1}{B_{N,N+a_N}b_N}|S'_{N+a_N}-S'_N|\geqslant(1-\delta)T_-\right\}$$

$$\geqslant\exp\left\{-\frac{(1+\delta)(1-\delta)^2T^2_-B^2_{N,N+a_N}b^2_N}{2v^2_{N,N+a_N}}\right\} \quad (2.4.22)$$

$$\geqslant\left(\frac{B^2_{a_N}}{B^2_{N+a_N}\log B^2_{a_N}}\right)^{1-\delta}\geqslant\frac{B^2_{a_N}}{B^2_{N+a_N}\log B^2_{N+a_N}}.$$

According to the definitions of $T_-$ and $T_+$, we can choose $\varepsilon$ such that the right-hand side of (2.4.22) is larger than

$$Rv^2_{N,N+a_N}/(v^2_{o,N+a_N}\log v^2_{o,N+a_N}) \quad (2.4.23)$$

for some $R>0$. Let $0<\eta<\delta\wedge(\delta^2T^2_-/4T^2_+)$, $N_1=1$. Define $N_{k+1}$ by

$$N_{k+1}=\min\{n:v^2_{o,n}+\eta v^2_{n,n+a_n}\geqslant v^2_{o,N_k+a_{N_k}}\}. \quad (2.4.24)$$

Then, we have

$$v^2_{o,N_{k+1}}+\eta v^2_{N_{k+1},N_{k+1}+a_{N_{k+1}}}\geqslant v^2_{o,N_k+a_{N_k}}\quad\text{for every } k, \quad (2.4.25)$$

$$v^2_{o,n}+\eta v^2_{n,n+a_n}<v^2_{o,N_k+a_{N_k}}\quad\text{for every } n<N_{k+1}. \quad (2.4.26)$$

Hence we find that $N_{k+1}>N_k$ and $N_{k+1}+a_{N_{k+1}}>N_k+a_{N_k}$ for every $k\geqslant 1$. At first, we prove that

$$\sum_{k=1}^{\infty}v^2_{N_k,N_k+a_{N_k}}/(v^2_{o,N_k+a_{N_k}}\log v^2_{o,N_k+a_{N_k}})=\infty. \quad (2.4.27)$$

In terms of (2.4.26), we get

$$v^2_{o,N_{k-1}+a_{N_{k-1}}}\geqslant v^2_{o,N_{k-1}}=v^2_{o,N_k}-\operatorname{Var}X'_{N_k} \quad (2.4.28)$$

$$\geqslant v^2_{o,N_k}-\varepsilon^2B'^2_{N_k}b'^{-2}_{N_k}\geqslant v^2_{o,N_k}-H\varepsilon^2B^2_{N_k,N_k+a_{N_k}}b'^{-2}_{N_k}$$

$$\geqslant v^2_{o,N_k}-v^2_{N_k,N_k+a_{N_k}}=v^2_{o,N_k+a_{N_k}}-2v^2_{N_k,N_k+a_{N_k}},$$

when $k$ is large enough. And

$$v_{o,N_{k-1}+a_{N_{k-1}}}^2 \geqslant \eta v_{o,N_k-1+a_{N_k-1}}^2 \geqslant \frac{\eta}{C} v_{o,N_k+a_{N_k}}^2 \qquad (2.4.29)$$

by (2.4.26) and (2.4.7). Now using (2.4.28) and (2.4.29), we have

$$\sum_{k=2}^{\infty} v_{N_k,N_k+a_{N_k}}^2 / (v_{o,N_k+a_{N_k}}^2 \log v_{o,N_k+a_{N_k}}^2)$$

$$\geqslant \frac{1}{2} \sum_{k=2}^{\infty} (v_{o,N_k+a_{N_k}}^2 - v_{o,N_{k-1}+a_{N_{k-1}}}^2) / (v_{o,N_k+a_{N_k}}^2 \log v_{o,N_k+a_{N_k}}^2)$$

$$\geqslant \frac{\eta}{2C} \sum_{k=2}^{\infty} (v_{o,N_k+a_{N_k}}^2 - v_{o,N_{k-1}+a_{N_{k-1}}}^2) / (v_{o,N_{k-1}+a_{N_{k-1}}}^2 \log(\frac{C}{\eta} v_{o,N_{k-1}+a_{N_{k-1}}}^2))$$

$$\geqslant \frac{\eta}{2C} \sum_{k=2}^{\infty} \int_{v_{o,N_{k-1}+a_{N_{k-1}}}^2}^{v_{o,N_k+a_{N_k}}^2} \frac{1}{x \log x} dx = \infty,$$

which proves that (2.4.27) holds true.

Put $G = \{ k : N_k \geqslant N_{k-1} + a_{N_{k-1}} \}$, $K = \{ k : N_k < N_{k-1} + a_{N_{k-1}} \}$. To prove the left part of (2.4.20), wc consider two cases:

*Case* 1 Suppose that

$$\sum_{k \in G} v_{N_k,N_k+a_{N_k}}^2 / (v_{o,N_k-a_{N_k}}^2 \log v_{o,N_k+a_{N_k}}^2) = \infty. \qquad (2.4.30)$$

Then, by (2.4.23)

$$\sum_{k \in G} P\{ \frac{1}{B_{N_k,N_k+a_{N_k}} b_{N_k}} | S'_{N_k+a_{N_k}} - S'_{N_k} | \geqslant (1-\delta) T_- \} = \infty. \qquad (2.4.31)$$

By noting that $\{ S'_{N_k+a_{N_k}} - S'_{N_k}, \ k \in G \}$ is an independent sequence, (2.4.31) implies that

$$\overline{\lim_{\substack{k \to \infty \\ k \in G}}} \frac{| S'_{N_k+a_{N_k}} - S'_{N_k} |}{B_{N_k,N_k+a_{N_k}} b_{N_k}} \geqslant (1-\delta) T_- \quad \text{a.s.}$$

and hence

$$\overline{\lim_{N \to \infty}} \frac{| S'_{N+a_N} - S'_N |}{B_{N,N+a_N} b_N} \geqslant (1-\delta) T_- \quad \text{a.s.}$$

i.e. the left part of (2.4.20) holds true.

*Case* 2 Suppose that

$$\sum_{k\in G} v^2_{N_k, N_k+a_{N_k}}/(v^2_{o, N_k+a_{N_k}} \log v^2_{o, N_k+a_{N_k}}) < \infty . \tag{2.4.32}$$

Then, by (2.4.27)

$$\sum_{k\in K} v^2_{N_k, N_k+a_{N_k}}/(v^2_{o, N_k+a_{N_k}} \log v^2_{o, N_k+a_{N_k}}) = \infty . \tag{2.4.33}$$

For every $k \in K$, by (2.4.25)

$$0 \leqslant v^2_{o, N_{k-1}+a_{N_{k-1}}} - v^2_{o, N_k} \leqslant \eta v^2_{N_k, N_k+a_{N_k}} . \tag{2.4.34}$$

Hence

$$(1-\eta) v^2_{N_k, N_k+a_{N_k}} \leqslant v^2_{o, N_k+a_{N_k}} - v^2_{o, N_{k-1}+a_{N_{k-1}}} \leqslant v^2_{N_k, N_k+a_{N_k}} . \tag{2.4.35}$$

Write

$$| S'_{N_k+a_{N_k}} - S'_{N_k} | \geqslant | S'_{N_k+a_{N_k}} - S'_{N_{k-1}+a_{N_{k-1}}} | - | S'_{N_{k-1}+a_{N_{k-1}}} - S'_{N_k} | .$$

Consider the first term of the right-hand side. Noting (2.4.35), we can use Lemma 2.2.1 (b) again like (2.4.22) and obtain that

$$\sum_{k\in K} P\left\{ \frac{1}{B_{N_k, N_k+a_{N_k}} b_{N_k}} | S'_{N_k+a_{N_k}} - S'_{N_{k-1}+a_{N_{k-1}}} | \geqslant (1-\delta) T_- \right\}$$

$$\geqslant \sum_{k\in K} (B^2_{a_{N_k}}/(B^2_{N_k+a_{N_k}} \log B^2_{N_k+a_{N_k}}))^{(1-\delta)/(1-\eta)}$$

$$\geqslant c \sum_{k\in K} v^2_{N_k, N_k+a_{N_k}} /(v^2_{o, N_k+a_{N_k}} \log v^2_{o, N_k+a_{N_k}}) = \infty ,$$

which implies that

$$\varlimsup_{\substack{k\to\infty \\ k\in K}} \frac{| S'_{N_k+a_{N_k}} - S'_{N_{k-1}+a_{N_{k-1}}} |}{B_{N_k, N_k+a_{N_k}} b_{N_k}} \geqslant (1-\delta) T_- . \tag{2.4.36}$$

By using (2.4.34) and noting $\eta \leqslant \delta^2 T_-^2 / 4T_+^2$ ,

$$\prod_{j=N_k-1}^{N_{k-1}+a_{N_{k-1}}} E \exp\left\{ \frac{3 b_{N_k}}{\delta T_- B_{N_k, N_k+a_{N_k}}} (X'_j - EX'_j) \right\}$$

$$\leqslant \exp\left\{ \frac{3b_{N_k}^2}{\delta^2 T_-^2 B_{N_k, N_k+a_{N_k}}^2} \sum_{j=N_{k+1}}^{N_{k-1}+a_{N_{k-1}}} \mathrm{Var} X_j' \right\}$$

$$\leqslant \exp\{ 3\eta b_{N_k}^2 v_{N_k, N_k+a_{N_k}}^2 / \delta^2 T_-^2 B_{N_k, N_k+a_{N_k}}^2 \}$$

$$\leqslant \exp\{ 4\eta T_+^2 b_{N_k}^2 / \delta^2 T_-^2 \}$$

$$\leqslant \exp(b_{N_k}^2)$$

for large $k$ and small $\varepsilon$. Hence, using the definitions of $T_+$ and $T_-$, for large $k$ and small $\varepsilon$, we have

$$P\left\{ \frac{1}{B_{N_k, N_k+a_{N_k}} b_{N_k}} | S'_{N_{k-1}+a_{N_{k-1}}} - S'_{N_k} | \geqslant \delta T_- \right\} \tag{2.4.37}$$

$$\leqslant \exp\{-3b_{N_k}^2 + b_{N_k}^2\} = B_{a_{N_k}}^4 / (B_{N_k+a_{N_k}}^4 \log^2 B_{a_{N_k}}^2)$$

$$\leqslant v_{N_k, N_k+a_{N_k}}^3 / (v_{o, N_k+a_{N_k}}^3 \log^2 v_{N_k, N_k+a_{N_k}}^2)$$

$$\leqslant v_{N_k, N_k+a_{N_k}}^2 / (v_{o, N_k+a_{N_k}}^2 \log^2 v_{o, N_k+a_{N_k}}^2).$$

By (2.4.25)

$$v_{N_k, N_k+a_{N_k}}^2 = v_{o, N_k+a_{N_k}}^2 - v_{o, N_k}^2 \leqslant v_{o, N_k+a_{N_k}}^2 - v_{o, N_{k-1}+a_{N_{k-1}}}^2 + \eta v_{N_k, N_k+a_{N_k}}^2 .$$

So, if $\eta < 1/2$,

$$v_{N_k, N_k+a_{N_k}}^2 \leqslant 2(v_{o, N_k+a_{N_k}}^2 - v_{o, N_{k-1}+a_{N_{k-1}}}^2).$$

Using this inequality and imitating the proof of (2.4.27), we can get

$$\sum_{k=1}^{\infty} v_{N_k, N_k+a_{N_k}}^2 / (v_{o, N_k+a_{N_k}}^2 \log^2 v_{o, N_k+a_{N_k}}^2) < \infty .$$

Therefore

$$\sum_{k \in K} v_{N_k, N_k+a_{N_k}}^2 / (v_{o, N_k+a_{N_k}}^2 \log^2 v_{o, N_k+a_{N_k}}^2) < \infty .$$

Thus, from (2.4.37), we have

$$\varlimsup_{\substack{k \to \infty \\ k \in K}} \frac{|S'_{N_{k-1}+a_{N_{k-1}}} - S'_{N_k}|}{B_{N_k, N_k+a_{N_k}} b_{N_k}} \leqslant \delta T_- \quad \text{a.s.} \tag{2.4.38}$$

Combining (2.4.38) with (2.4.36) yields

$$\varlimsup_{\substack{k \to \infty \\ k \in K}} \frac{|S'_{N_k+a_{N_k}} - S'_{N_k}|}{B_{N_k, N_k+a_{N_k}} b_{N_k}} \geqslant (1-2\delta) T_- \quad \text{a.s.}$$

and hence

$$\overline{\lim_{N\to\infty}} \frac{|S'_{N+a_N}-S'_N|}{B_{N,N+a_N} b_N} \geqslant (1-2\delta)T_- \qquad \text{a.s.}$$

i.e. the left part of (2.4.20) holds true. The theorem is proved.

## 2.5 How Small Are the Increments of Partial Sums of Independent R. V.'s?

Let $\{X_n, n\geqslant 1\}$ be a sequence of i. i. d. r. v. 's with $EX_1=0$ and $EX_1^2=1$. Chung (1948) was the first one who discussed the problem of the inferior limit of $\max\limits_{1\leqslant i\leqslant n}|S_i|$ and proved

$$\underline{\lim_{n\to\infty}} \left\{\frac{8\log\log n}{\pi^2 n}\right\}^{1/2} \max_{1\leqslant i\leqslant n}|S_i|=1 \qquad \text{a.s.} \tag{2.5.1}$$

Csáki (1978) showed that the converse of (2. 5. 1) is also true. Csörgő and Révész (1981) considered the increments of partial sums and gave Theorem 2.1.4. As we have pointed out in Section 2. 1., the proof of this theorem cannot carry conviction. In this section, not only is its strict proof given but also this is extended to the non-identically distributed case under weaker assumptions.

**Theorem 2.5.1**( Shao 1989) *Let $\{X_n;\ n\geqslant 1\}$ be a sequence of independent random variables with $EX_n=0$ and $EX_n^2\geqslant C_0>0$ for every $n\geqslant 1$. Suppose that $\{X_n^2;\ n\geqslant 1\}$ is uniformly integrable. Let $\{a_N;N\geqslant 1\}$ be a sequence of integers satisfying the following conditions:*

(i) $1\leqslant a_N\leqslant N$,

(ii) $a_N/\log N\to\infty$ as $N\to\infty$.

*Then, putting $\sigma_{nk}^2=\sum\limits_{i=n+1}^{n+k} EX_i^2$, $S_0=0$, we have*

$$\underline{\lim_{N\to\infty}} \min_{0\leqslant n\leqslant N} \max_{1\leqslant k\leqslant a_N} \gamma_{nN}|S_{n+k}-S_n|=1 \qquad \text{a.s.} \tag{2.5.2}$$

$$\varliminf_{N \to \infty} \max_{1 \leqslant k \leqslant a_N} \gamma_{NN} | S_{N+k} - S_N | = 1 \qquad \text{a.s.} \tag{2.5.3}$$

*where* $\gamma_{nN} = \{ 8(\log N/a_N + \log\log N)/\pi^2 \sigma^2_{na_N} \}^{1/2}$.

*If, in addition, we have*

(iii) $\lim_{N \to \infty} (\log N/a_N)/\log\log N = \infty$,

*then*

$$\lim_{N \to \infty} \min_{0 \leqslant n \leqslant N} \max_{1 \leqslant k \leqslant a_N} \gamma_{nN} | S_{n+k} - S_n | = 1 \qquad \text{a.s.} \tag{2.5.4}$$

In order to prove this theorem, we need the following lemmas.

**Lemma 2.5.1** *Let* $\{ X_n \}$ *be a sequence of independent random variables. Assume that there exist* $\varepsilon > 0$, $0 < \alpha < 1$ *and an integer* $p \geqslant 1$ *such that*

$$P\{ \max_{1 \leqslant k \leqslant p} | S_k | \geqslant \varepsilon x \} \leqslant \alpha \tag{2.5.5}$$

*for some* $x > 0$. *Then*

$$P\{ \bigcup_{n=0}^{p} ( \max_{1 \leqslant k \leqslant N} | S_{n+k} - S_n | \leqslant x ) \} \leqslant \frac{1}{1-\alpha} P\{ \max_{1 \leqslant k \leqslant N} | S_k | \leqslant (1+\varepsilon) x \}. \tag{2.5.6}$$

*Proof* Put

$$E_p = \{ \max_{1 \leqslant k \leqslant N} | S_{p+k} - S_p | \leqslant x \},$$

$$E_i = \bigcap_{i < n \leqslant p} \{ \max_{1 \leqslant k \leqslant N} | S_{n+k} - S_n | > x \} \bigcap \{ \max_{1 \leqslant k \leqslant N} | S_{i+k} - S_i | \leqslant x \},$$

$$i = p-1, p-2, \cdots, 0.$$

Obviously

$$\bigcup_{n=0}^{p} \{ \max_{1 \leqslant k \leqslant N} | S_{n+k} - S_n | \leqslant x \} = \bigcup_{n=0}^{p} E_n$$

$$\subset \{ \max_{1 \leqslant k \leqslant N} | S_k | < (1+\varepsilon) x \} \bigcup ( \bigcup_{n=1}^{p} (E_n \bigcap \{ \max_{1 \leqslant k \leqslant N} | S_k | \geqslant (1+\varepsilon) x \}))$$

$$\subset \{ \max_{1 \leqslant k \leqslant N} | S_k | < (1+\varepsilon) x \} \bigcup ( \bigcup_{n=1}^{p} (E_n \bigcap \{ \max_{1 \leqslant k \leqslant n} | S_k | \geqslant (1+\varepsilon) x \}))$$

$$\bigcup ( \bigcup_{n=1}^{p} (E_n \bigcap \{ \max_{n < k \leqslant N} | S_k | \geqslant (1+\varepsilon) x \}))$$

$$\subset \{ \max_{1 \leqslant k \leqslant N} |S_k| < (1+\varepsilon)x \} \bigcup ( \bigcup_{n=1}^{p} (E_n \bigcap \{ \max_{1 \leqslant k \leqslant n} |S_k| \geqslant (1+\varepsilon)x \}))$$

$$\bigcup ( \bigcup_{n=1}^{p} (E_n \bigcap \{ |S_n| \geqslant \varepsilon x \} ))$$

$$\subset \{ \max_{1 \leqslant k \leqslant N} |S_k| < (1+\varepsilon)x \} \bigcup ( \bigcup_{n=1}^{p} ( E_n \bigcap \{ \max_{1 \leqslant k \leqslant n} |S_k| \geqslant \varepsilon x \})).$$

Noting that $E_n$ and $\{ \max_{1 \leqslant k \leqslant n} |S_k| \geqslant \varepsilon x \}$ are independent, we have

$$P\{ \bigcup_{n=0}^{p} (\max_{1 \leqslant k \leqslant N} |S_{n+k} - S_n| \leqslant x )\}$$

$$\leqslant P\{ \max_{1 \leqslant k \leqslant N} |S_k| \leqslant (1+\varepsilon)x \} + \sum_{n=1}^{p} P(E_n) P\{ \max_{1 \leqslant k \leqslant n} |S_k| \geqslant \varepsilon x \}$$

$$\leqslant P\{ \max_{1 \leqslant k \leqslant N} |S_k| \leqslant (1+\varepsilon)x \} + \alpha \sum_{n=1}^{p} P(E_n)$$

$$\leqslant P\{ \max_{1 \leqslant k \leqslant N} |S_k| \leqslant (1+\varepsilon)x \} + \alpha P\{ \bigcup_{n=0}^{p} (\max_{1 \leqslant k \leqslant N} |S_{n+k} - S_n| \leqslant x )\}$$

as desired.

**Lemma 2.5.2** *Let $\{X_n\}$ be a sequence of random variables satisfying the conditions of Theorem* 2.5.1. *Then*

$$\log P\{ \max_{1 \leqslant k \leqslant a_N} |S_{n+k} - S_n| \leqslant x_{nN} \sigma_{n a_N} \} \sim -\frac{\pi^2}{8 x_{nN}^2}$$

*uniformly in* $1 \leqslant n \leqslant N$, *provided that* $\max_{1 \leqslant n \leqslant N} x_{nN} \to 0$ *and* $\min_{1 \leqslant n \leqslant N} x_{nN} \sigma_{na_N} \to \infty$ *as* $N \to \infty$.

The proof of Lemma 2.5.2 can be found in Shao (1989).

Proof of Theorem 2.5.1

The proof is formulated in three steps, which together will imply the statements of the theorem.

*Step* 1 For any $0 < \varepsilon < 1/4$, we have

$$\varliminf_{N \to \infty} \min_{0 \leqslant n \leqslant N} \max_{1 \leqslant k \leqslant a_N} \gamma_{nN} |S_{n+k} - S_n| \geqslant 1 - 3\varepsilon \quad \text{a.s.} \qquad (2.5.7)$$

*Proof* From the uniform integrability of $\{ X_n^2 \}$, there is a constant $C_2 > 0$ such that

$$C_0 \leqslant EX_n^2 \leqslant C_2 \tag{2.5.8}$$

for every $n \geqslant 1$. Let

$$1 < \theta < 1 + \varepsilon C_0 / 4C_2 . \tag{2.5.9}$$

Define

$$H_k = \{ N : [\theta^k] < a_N \leqslant [\theta^{k+1}] \},$$

$$M_k = \max \{ N : N \in H_k \},$$

$$D_L = [\log_\theta ( \inf_{N \geqslant L} a_N )] - 1.$$

Note that

$$\inf_{N \geqslant L} \min_{0 \leqslant n \leqslant N} \max_{1 \leqslant i \leqslant a_N} \gamma_{nN} | S_{n+i} - S_n | \tag{2.5.10}$$

$$\geqslant \inf_{k \geqslant D_L} \min_{N \in H_k} \min_{0 \leqslant n \leqslant N} \max_{1 \leqslant i \leqslant a_N} \gamma_{nN} | S_{n+i} - S_n |$$

$$\geqslant \inf_{k \geqslant D_L} \min_{0 \leqslant n \leqslant M_k} \max_{1 \leqslant i \leqslant \theta^k} \gamma'_{nk} | S_{n+i} - S_n | ,$$

where $\gamma'_{nk} = \{ 8 ( \log ( n \vee \theta^{k+1} ) / \theta^{k+1} + \log \log \theta^{k+1} ) / \pi^2 \sigma^2_{n, [\theta^{k+1}]} \}^{1/2}$. Let $p_0 = 0, p_j = [ C_0 \varepsilon^3 \theta^k / 32 C_2 \log ( jk )], j = 1, \cdots, m_k := \min \{ n : \sum_{j=0}^{n} p_j \geqslant M_k \}$. It is easy to see that

$$q_j := \sum_{i=0}^{j} p_i \sim j C_0 \varepsilon^3 \theta^h / 32 C_2 \log ( jk ). \tag{2.5.11}$$

From (2.5.8), we have

$$\lim_{k \to \infty} \max_{0 \leqslant j < M_k} \max_{q_j \leqslant n \leqslant q_{j+1}} \gamma'_{q_j k} / \gamma'_{nk} = 1,$$

which together with Levy's maximal inequality implies

$$P \{ \max_{0 \leqslant i \leqslant p_{j+1}} | S_{q_j + i} - S_{q_j} | \geqslant \varepsilon \min_{q_j \leqslant n \leqslant q_{j+1}} 1 / \gamma'_{nk} \}$$

$$\leqslant P \{ \max_{0 \leqslant i \leqslant p_{j+1}} | S_{q_j + i} - S_{q_j} | \geqslant \varepsilon / 2 \gamma'_{q_j k} \}$$

$$\leqslant 2P \{ | S_{q_j + p_{j+1}} - S_{q_j} | \geqslant \varepsilon / 4 \gamma'_{q_j k} \}$$

$$\leqslant 32 C_2 p_{j+1} \gamma'^2_{q_j k} / \varepsilon^2 \leqslant 1/2$$

for large $k$. Hence, using Lemma 2.5.1, we have

$$P\{\min_{0\leqslant n\leqslant M_k}\max_{1\leqslant i\leqslant\theta^k}\gamma'_{nk}|S_{n+i}-S_n|\leqslant 1-3\varepsilon\}$$

$$\leqslant\sum_{j=0}^{m_k-1}P\{\min_{q_j\leqslant n\leqslant q_{j+1}}\max_{1\leqslant i\leqslant\theta^k}\gamma'_{nk}|S_{n+i}-S_n|\leqslant 1-3\varepsilon\} \tag{2.5.12}$$

$$\leqslant 2\sum_{j=0}^{m_k-1}P\{\max_{1\leqslant i\leqslant\theta^k}\gamma'_{q_jk}|S_{q_j+i}-S_{q_j}|\leqslant 1-\varepsilon\}.$$

Applying Lemma 2.5.2. and noting (2.5.9), we obtain

$$P\{\max_{1\leqslant i\leqslant\theta^k}\gamma'_{q_jk}|S_{q_j+i}-S_{q_j}|\leqslant 1-\varepsilon\} \tag{2.5.13}$$

$$\leqslant\exp\{-\frac{\pi^2}{8(1-\varepsilon)}\sigma^2_{q_j,[\theta^k]}\gamma'^2_{q_jk}\}$$

$$\leqslant\exp\{-(1-\varepsilon)^{-1/2}(\log(q_j\vee\theta^{k+1})/\theta^{k+1}+\log\log\theta^{k+1})\}$$

$$\leqslant\{((q_j\vee\theta^{k+1})/\theta^{k+1})\log\theta^{k+1}\}^{-(1-\varepsilon)^{-1/2}}.$$

By (2.5.11), (2.5.12) and (2.5.13), we have

$$\sum_{k=1}^{\infty}P\{\min_{0\leqslant n\leqslant M_k}\max_{1\leqslant i\leqslant\theta^k}\gamma'_{nk}|S_{n+i}-S_n|\leqslant 1-3\varepsilon\}$$

$$\leqslant 2\sum_{k=1}^{\infty}\sum_{j=0}^{m_k-1}\{((q_j\vee\theta^{k+1})/\theta^{k+1})\log\theta^{k+1}\}^{-(1-\varepsilon)^{-1/2}}$$

$$\leqslant C(\theta)\sum_{k=1}^{\infty}\{(\log k)k^{-(1-\varepsilon)^{-1/2}}+\sum_{j=0}^{m_k-1}(jk/\log(jk))^{-(1-\varepsilon)^{-1/2}}\}$$

$$\leqslant 2C(\theta)\{\sum_{k=1}^{\infty}((\log k)/k)^{(1-\varepsilon)^{-1/2}}\}^2$$

$$\leqslant C(\theta,\varepsilon)$$

for some $C(\theta)>0$ and $C(\theta,\varepsilon)>0$. This implies that

$$\varliminf_{k\to\infty}\min_{0\leqslant n\leqslant M_k}\max_{1\leqslant i\leqslant\theta^k}\gamma'_{nk}|S_{n+i}-S_n|\geqslant 1-3\varepsilon\qquad\text{a.s.}$$

(2.5.7) is proved by (2.5.10).

*Step* 2 For any $0<\varepsilon<1/4$, we have

$$\varlimsup_{N\to\infty}\max_{1\leqslant k\leqslant a_N}\gamma_{NN}|S_{N+k}-S_N|\leqslant 1/(1-\varepsilon)\qquad\text{a.s.} \tag{2.5.14}$$

*Proof* Let $N_1=1$. Define $N_{k+1}=N_k+a_{N_k}$, $k\geqslant 1$. Using Lemma 2.5.2 again, we find

$$\begin{aligned} & P\{\max_{0\leqslant n\leqslant a_{N_k}} \gamma_{N_k N_k} | S_{N_k+n}-S_{N_k}| \leqslant 1/(1-\varepsilon)\} \\ & \geqslant \exp(-(1-\varepsilon)(\log N_k/a_{N_k}+\log\log N_k)\} \qquad (2.5.15) \\ & \geqslant (N_{k+1}-N_k)/(N_k \log N_k) \end{aligned}$$

which together with the Borel-Cantelli lemma implies

$$\varliminf_{k\to\infty} \max_{0\leqslant n\leqslant a_{N_k}} \gamma_{N_k N_k} | S_{N_k+n}-S_{N_k}| \leqslant 1/(1-\varepsilon) \quad \text{a.s.} \qquad (2.5.16)$$

since $\max_{0\leqslant n\leqslant a_{N_k}} \gamma_{N_k N_k} | S_{N_k+n}-S_{N_k}|$, $k=1, 2, \cdots$, are independent and $\sum_{k=1}^{\infty}(N_{k+1}-N_k)/(N_k\log N_k)=\infty$. This proves (2.5.14).

*Step* 3 If condition (iii) is satisfied, then for any $0<\varepsilon<1/4$ we have

$$\varlimsup_{N\to\infty} \min_{0\leqslant n\leqslant N} \max_{1\leqslant k\leqslant a_N} \gamma_{nN} | S_{n+k}-S_n| \leqslant 1+\varepsilon \quad \text{a.s.} \qquad (2.5.17)$$

*Proof* It is easy to see that $\lim_{N\to\infty} N/a_N=\infty$ by Condition (iii). Let $p:=p_N=[N/a_N]$. Then, from Lemma 2.5.2,

$$\begin{aligned} & P\{\min_{0\leqslant n\leqslant N} \max_{1\leqslant k\leqslant a_N} \gamma_{nN} | S_{n+k}-S_n| \geqslant 1+\varepsilon\} \\ & \leqslant P\{\min_{0\leqslant j\leqslant p} \max_{1\leqslant k\leqslant a_N} \gamma_{ja_N, N} | S_{ja_N+k}-S_{ja_N}| \geqslant 1+\varepsilon\} \\ & = \prod_{j=0}^{p} \{1-P(\max_{1\leqslant k\leqslant a_N} \gamma_{ja_N, N} | S_{ja_N+k}-S_{ja_N}| <1+\varepsilon)\} \qquad (2.5.18) \\ & \leqslant \prod_{j=0}^{p} \{1-\exp\{-\frac{1}{1+\varepsilon}\log\frac{N\log N}{a_N}\}\} \\ & \leqslant \prod_{j=0}^{p} \{1-(a_N/N)^{1/(1+\varepsilon)}/\log N\} \\ & \leqslant \exp\{-\sum_{j=0}^{p}(a_N/N)^{1/(1+\varepsilon)}/\log N\} \end{aligned}$$

$$\leqslant \exp\{-(N/a_N)^{\varepsilon/2}/\log N\}$$

$$\leqslant N^{-2}$$

for large enough $N$, where the last inequality is due to Condition (iii). Using the Borel-Cantelli lemma again, we obtain (2.5.17). This completes the proof of Theorem 2.5.1.

From Theorem 2.5.1, we have the following corollary immediately.

**Corollary 2.5.1** (Shao 1989) *Let $\{X_n\}$ be a sequence of independent random variables with $EX_n=0$. Suppose that $\{X_n^2;\ n\geqslant 1\}$ is uniformly integrable and*

$$D_N=\sum_{i=1}^{N} EX_i^2 \to \infty,\ N\to\infty.$$

*Then we have*

$$\varliminf_{N\to\infty}\left(\frac{8\log\log D_N}{\pi^2 D_N}\right)^{1/2}\max_{1\leqslant n\leqslant N}|S_n|=1 \quad \text{a.s.} \qquad (2.5.19)$$

*Remark* 2.5.1 There are some more general conclusions in Shao (1989). He has also given an example for which the law of the iterated logarithm fails but the Chung's law of the iterated logarithm holds true.

*Example* 2.5.1 Let $\{X_n;\ n\geqslant 1\}$ be a sequence of independent random variables with distribution:

$$P\{X_n=n^{1/2}\log\log n\}=P\{X_n=-n^{1/2}\log\log n\}=(2n(\log\log n)^3)^{-1},$$

$$P\{X_n=2\}=P\{X_n=-2\}=(1-1/\log\log n)/8,$$

$$P\{X_n=0\}=\frac{3}{4}-\frac{1}{n(\log\log n)^3}+\frac{1}{4\log\log n}.$$

Clearly, $EX_n=0$, $EX_n^2=1$ and $\{X_n^2;n\geqslant 1\}$ is uniformly integrable. It follows from Corollary 2.5.1 that

$$\varliminf_{N\to\infty}\max_{1\leqslant n\leqslant N}\left(\frac{8\log\log N}{\pi^2 N}\right)^{1/2}\left|\sum_{i=1}^{n} X_i\right|=1 \quad \text{a.s.}$$

On the other hand, for any $c>0$

$$\sum_{n=1}^{\infty} P\{\,|X_n| \geqslant c\,(n \log\log n)^{1/2}\,\} = \infty\ .$$

Thus

$$\overline{\lim_{N\to\infty}}\ \left|\sum_{i=1}^{N} X_i\right| \Big/ (2N \log\log N)^{1/2} = \infty \qquad \text{a.s.}$$

That is to say, even though the law of the iterated logarithm fails, Chung's law of the iterated logarithm is still true. But we don't know, if there is a sequence of independent random variables satisfying the law of the iterated logarithm and not satisfying Chung's law of the iterated logarithm. We Still do not know how to describe the necessary conditions for Chung's law of the iterated logarithm.

## 2.6 A Study of Partial Sums with the Help of Strong Approximation

We have established some results on the increments of partial sums in the previous sections, via estimating the probability of partial sums directly. On the other hand, using the strong approximation, one can easily obtain the properties of increments of partial sums for i.i.d. random variables (c.f. Csörgő-Révész 1981). In this section, we first derive the strong approximation theorems for independent random variables from the Sakhanenko (1984) results, and then consider the increments of partial sums.

### 2.6.1 Sakhanenko's Theorem

The strong approximation theorem is one of siginificant achievements in the development of modern probability theory. It says that the partial sums of random variables can be approximated by a Wiener process. If the approximation error is sufficiently small, then many of the limit theorems known

and usually more easily proved for a Wiener process will continue to hold for a the partial sum process under consideration. For i.i.d. random variables, the best possible results are due to Komlós-Major-Tusnády (1975, 1976).

**Theorem 2. 6. 1** (Komlós, Major, Tusnády 1975, 1976) *Let* $\{X_n; n\geqslant 1\}$ *be a sequence of* i.i.d. *random variables with* $EX_1=0$, $EX_1^2=1$, $E|X_1|^p<\infty$. *Then, we can redefine* $\{X_n; n\geqslant 1\}$ *on a richer probability space together with a standard Wiener process* $\{W(t);\ t\geqslant 0\}$ *such that*

$$\sum_{i=1}^{n} X_i - W(n) = o(n^{1/p}) \qquad \text{a.s.} \tag{2.6.1}$$

For general results on the strong approximation and their applications, the reader can refer to Csörgő-Révész (1981).

For independent non-identically distributed random variables, Sakhanenko (1984) established the following profound result.

**Theorem 2.6.2** (Sakhanenko 1984) *Let* $\{\xi_n; n\geqslant 1\}$ *be a sequence of independent random variables with* $E\xi_n=0$ *and for any* $j\geqslant 1$

$$\lambda E|\xi_j|^3 e^{\lambda|\xi_j|} \leqslant E\xi_j^2 \ \text{ for some } \lambda>0\,. \tag{2.6.2}$$

*Then, we can redefine* $\{\xi_n; n\geqslant 1\}$ *on a richer probability space together with a sequence of independent normal random variables* $\{\eta_n;\ n\geqslant 1\}$ *with* $\eta_n \sim N(0, \mathrm{Var}\xi_n)$ *such that*

$$E\exp\{C\lambda \max_{i\leqslant n}|\sum_{j=1}^{i}\xi_j-\sum_{j=1}^{i}\eta_j|\}\leqslant 1+\lambda\sum_{i=1}^{n}\mathrm{Var}\xi_i \tag{2.6.3}$$

*where* $C$ *is an absolute constant.*

The proof of Theorem 2.6.2 is too complicated to state it here. From Theorem 2.6.2, we have

**Theorem 2.6.3** *Suppose that* (2.6.2) *is satisfied and that*

$$\sum_{i=1}^{n} E\xi_i^2 \leqslant B_n \to \infty \quad \text{as} \quad n\to\infty\,,$$

*where* $\{B_n\}$ *is a sequence of positive numbers. Then, we have*

$$|\sum_{i=1}^{n}\xi_i-\sum_{i=1}^{n}\eta_i|\leqslant \frac{1}{\lambda C}\log B_n \qquad \text{a.s.} \tag{2.6.4}$$

*In particular, if* $\sum_{i=1}^{n} E\xi_i^2 \to \infty$, *then*

$$|\sum_{i=1}^{n}\xi_i-\sum_{i=1}^{n}\eta_i|\leqslant \frac{1}{\lambda C}\log(\sum_{i=1}^{n}\text{Var}\xi_i) \quad \text{a.s.} \tag{2.6.5}$$

*Proof* Put $A_k=\{n: 2^k<B_n\leqslant 2^{k+1}\}$, $\mathscr{A}=\{k; A_k\neq\varPhi\}$. Noting that $B_n \to \infty$, we have that $\mathscr{A}$ contains infinite numbers. Write $\mathscr{A}=\{k_1, k_2, \cdots\}$, $n_i=\max\{m: m\in A_{k_i}\}$. Then, for each $\varepsilon>0$,

$$\begin{aligned}
&P\{\max_{i\leqslant nk}|\sum_{j=1}^{i}\xi_j-\sum_{j=1}^{i}\eta_j|\geqslant\frac{1+\varepsilon}{\lambda C}\log B_{nk}\}\\
&\leqslant\exp\{-(1+\varepsilon)\log B_{n_k}\}\cdot E\exp\{\lambda C\max_{i\leqslant n_k}|\sum_{j=1}^{i}\xi_j-\sum_{j=1}^{i}\eta_j|\}\\
&\leqslant\exp\{-(1+\varepsilon)\log B_{n_k}\}\cdot(1+\lambda\sum_{i=1}^{nk}\text{Var}\xi_i)\\
&\leqslant(1+\lambda B_{nk})\exp\{-(1+\varepsilon)\log B_{nk}\}\\
&\leqslant(1+\lambda B_{nk})/B_{nk}^{1+\varepsilon}\\
&\leqslant(1+\lambda)2^{-k\varepsilon}.
\end{aligned}$$

Thus, it follows from the Borel-Cantelli lemma that

$$\max_{i\leqslant nk}|\sum_{j=1}^{i}\xi_j-\sum_{j=1}^{i}\eta_j|\leqslant\frac{1+\varepsilon}{\lambda C}\log B_{n_k} \quad \text{a.s.} \tag{2.6.6}$$

For any $m\geqslant 1$, we can find an integer $k_i$ such that $m\in A_{k_i}$, and hence

$$\begin{aligned}
\max_{j\leqslant m}\frac{\left|\sum_{l=1}^{j}\xi_l-\sum_{l=1}^{j}\eta_l\right|}{\log B_m}&\leqslant\frac{\max_{j\leqslant n_i}\left|\sum_{l=1}^{j}\xi_l-\sum_{l=1}^{j}\eta_l\right|}{\log 2^{k_i}}\\
&\leqslant\frac{\log 2^{k_i+1}}{\log 2^{k_i}}\cdot\frac{\max_{j\leqslant n_i}\left|\sum_{l=1}^{j}\xi_l-\sum_{l=1}^{j}\eta_l\right|}{\log B_{n_i}}\\
&\leqslant\frac{(1+\varepsilon)}{\lambda C}\ \frac{\log 2^{k_i+1}}{\log 2^{k_i}}\\
&\leqslant\frac{1+2\varepsilon}{\lambda C}\qquad\text{a.s.}
\end{aligned}\tag{2.6.7}$$

provided that $m$ is sufficiently large. This proves (2.6.4) by (2.6.7) and the arbitrariness of $\varepsilon$.

**Theorem 2.6.4** *Let $\{H(n);n\geqslant 1\}$ be a non-decreasing sequence of positive numbers and $\{X_n;n\geqslant 1\}$ be a sequence of independent random variables with $EX_n=0$, $EX_n^2<\infty$. Assume that there exist $0<\alpha<\delta\leqslant 1, C>0$ such that*

$$\sum_{n=1}^{\infty} P\{|X_n|\geqslant \varepsilon\, H(n)\}<\infty \quad \textit{for every } n\geqslant 1. \tag{2.6.8}$$

$$E|X_n|^{2+\delta}\leqslant CH(n)^{\delta-\alpha}EX_n^2 \quad \textit{for every } n\geqslant 1, \tag{2.6.9}$$

$$n^\alpha\leqslant CH(n) \quad \textit{for any } n\geqslant 1. \tag{2.6.10}$$

*Then, we can redefine $\{X_n;n\geqslant 1\}$ on a richer probability space together with a sequence of independent normal random variables $\{Y_n;n\geqslant 1\}$ with $Y_n\sim N(0, \mathrm{Var}X_nI(|X_n|\leqslant H(n))$ such that*

$$\left|\sum_{i=1}^{n}(X_i-EX_iI(|X_i|\leqslant H(i)))-\sum_{i=1}^{n}Y_i\right|=o(H(n)) \qquad \text{a.s.} \tag{2.6.11}$$

*Proof* By (2.6.8), there exists $\varepsilon_n\to 0$ such that $\varepsilon_n\log n\to\infty$ and

$$\sum_{n=1}^{\infty}P\{|X_n|\geqslant\varepsilon_nH(n)\}<\infty. \tag{2.6.12}$$

Put $\overline{X}_n=X_nI(|X_n|\leqslant\varepsilon_nH(n))-EX_nI(|X_n|\leqslant\varepsilon_nH(n))$, $a_n=\max_{i\leqslant n}\varepsilon_i H(i)/\log H(i)$, $\xi_n=\overline{X}_n/a_n$. We first prove that for $\{\xi_n;n\geqslant 1\}$ (2.6.2) is satisfied. It suffices to show that for $0<\lambda<\alpha/16$

$$\lambda Ee^{\lambda|\overline{X}_n|/a_n}|\overline{X}_n|^3/a_n\leqslant E\overline{X}_n^2. \tag{2.6.13}$$

Noting that $x^\delta e^x$ is increasing on $(0,\infty)$ and that $a_n\geqslant\varepsilon_nH(n)/\log H(n)$, we have

$$\begin{aligned}
&\lambda Ee^{\lambda|\overline{X}_n|/a_n}|\overline{X}_n|^3/a_n\leqslant\lambda^\delta Ee^{2\lambda|\overline{X}_n|/a_n}|\overline{X}_n|^{2+\delta}/a_n^\delta\\
&\leqslant\lambda^\delta Ee^{2\lambda|\overline{X}_n|(\log H(n))/\varepsilon_nH(n)}|\overline{X}_n|^{2+\delta}/a_n^\delta\\
&\leqslant 9\lambda^\delta e^{4\lambda\log H(n)}E|X_n|^{2+\delta}/a_n^\delta\\
&\leqslant 9\lambda^\delta H(n)^{4\lambda}(\log H(n))^\delta E|X_n|^{2+\delta}/(\varepsilon_nH(n))^\delta
\end{aligned} \tag{2.6.14}$$

$$\leqslant 9C\lambda^{\delta}H(n)^{4\lambda}\, H(n)^{\delta-\alpha}EX_n^2(\log H(n))^{\delta}/(\varepsilon_n H(n))^{\delta}$$

$$\leqslant 9C\lambda^{\delta}H(n)^{-\frac{\alpha}{2}}\, EX_n^2$$

$$\leqslant EX_n^2/2$$

by (2.6.9) provided that $n$ is large enough. On the other hand, we have

$$E\overline{X}_n^2 = EX_n^2 I(|X_n| \leqslant \varepsilon_n H(n)) - (EX_n I(|X_n| > \varepsilon_n H(n)))^2 \tag{2.6.15}$$

$$\geqslant EX_n^2 - 2EX_n^2 I(|X_n| > \varepsilon_n H(n))$$

$$\geqslant EX_n^2 - 2E|X_n|^{2+\delta}/(\varepsilon_n H(n))^{\delta}$$

$$\geqslant EX_n^2 - 2CH(n)^{\delta-\alpha}EX_n^2/(\varepsilon_n H(n))^{\delta}$$

$$\geqslant EX_n^2(1 - 2C/(\varepsilon_n^{\delta} H(n)^{\alpha})$$

$$\geqslant EX_n^2/2$$

by (2.6.9) again, provided that $n$ is large enough. Combining (2.6.14) and (2.6.15) yields (2.6.13). By Theorem 2.6.3, we can redefine $\{\xi_n; n \geqslant 1\}$ on a richer probability space together with a sequence of independent normal random variables $\{\eta_n; n \geqslant 1\}$ with $\eta_n \sim N(0, \mathrm{Var}\xi_n)$ such that

$$\left|\sum_{i=1}^{n} \xi_i - \sum_{i=1}^{n} \eta_i\right| \leqslant \frac{16}{\alpha C} \log\left(\sum_{i=1}^{n} \mathrm{Var}\xi_i + H(n)\right) \quad \text{a.s.} \tag{2.6.16}$$

By (2.6.9), we get

$$(EX_n^2)^{\frac{2+\delta}{2}} \leqslant E|X_n|^{2+\delta} \leqslant C\, H(n)^{\delta-\alpha}EX_n^2.$$

That is

$$EX_n^2 \leqslant C^{2/\delta}\, H(n)^{2-2\alpha/\delta},$$

and hence

$$\sum_{i=1}^{n} \mathrm{Var}\xi_i + H(n) \leqslant 2\sum_{i=1}^{n} EX_i^2/a_i^2 + H(n)$$

$$\leqslant 2\sum_{i=1}^{n} (EX_i^2/H(i)^2)\log^2 H(i) + H(n)$$

$$\leqslant 2n\, c^{2/\delta}\, H(n)^2 + H(n)$$

$$\leqslant 3C^{2/\delta+1/\alpha}H(n)^{2+1/\alpha}$$

by (2.6.10), which together with (2.6.16) implies

$$|\sum_{i=1}^{n}\xi_i-\sum_{i=1}^{n}\eta_i|\leqslant\frac{16}{\alpha C}(2+1/\alpha)\log H(n)\qquad\text{a.s.}\tag{2.6.17}$$

Put $\overline{S}_n=\sum_{i=1}^{n}\overline{X}_i$, $T_n=\sum_{i=1}^{n}\overline{X}_i/a_i=\sum_{i=1}^{n}\xi_i$, $\overline{U}_n=\sum_{i=1}^{n}a_i\eta_i$, $V_n=\sum_{i=1}^{n}\eta_i$. Then

$$\begin{aligned}|\overline{S}_n-\overline{U}_n| &= |\sum_{i=1}^{n}a_i(T_i-T_{i-1})-\sum_{i=1}^{n}a_i(V_i-V_{i-1})|\\ &= |a_n(T_n-V_n)-\sum_{i=1}^{n-1}(T_i-V_i)(a_{i+1}-a_i)|\\ &\leqslant a_n|T_n-V_n|+a_n\max_{1\leqslant i\leqslant n}|T_i-V_i|\\ &= a_n\frac{32}{\alpha C}(2+1/\alpha)\log H(n)\qquad\text{a.s.}\end{aligned}$$

It is easy to see that $a_n\log H(n)=o(H(n))$. So we obtain

$$|\overline{S}_n-\overline{U}_n|=o(H(n))\qquad\text{a.s.}\tag{2.6.18}$$

Let $\overline{Y}_n=a_n\eta_n$, then $\{\overline{Y}_n; n\geqslant 1\}$ are independent normal random variables with $\overline{Y}_n\sim N(0,\mathrm{Var}X_nI(|X_n|\leqslant\varepsilon_nH(n)))$. In order to complete our proof , we set

$$Y_n=\frac{(\mathrm{Var}X_nI(|X_n|\leqslant H(n)))^{1/2}}{(\mathrm{Var}X_nI(|X_n|\leqslant\varepsilon_nH(n)))^{1/2}}\overline{Y}_n.$$

Clearly, $\{Y_n;\ n\geqslant 1\}$ are independent normal random variables with $Y_n\sim N(0,\mathrm{Var}X_nI(|X_n|\leqslant H(n)))$. Then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mathrm{Var}(Y_n-\overline{Y}_n)}{H(n)^2} &= \sum_{n=1}^{\infty}\frac{1}{H(n)^2}((\mathrm{Var}X_nI(|X_n|\leqslant H(n)))^{1/2}\\ &\quad-(\mathrm{Var}X_nI(|X_n|\leqslant\varepsilon_nH(n)))^{1/2})^2\\ &\leqslant\sum_{n=1}^{\infty}\frac{1}{H(n)^2}|\mathrm{Var}X_nI(|X_n|\leqslant H(n))-\mathrm{Var}X_nI(|X_n|\leqslant\varepsilon_nH(n))|\\ &\leqslant\sum_{n=1}^{\infty}\frac{1}{H(n)^2}(EX_n^2I(\varepsilon_nH(n)<|X_n|\leqslant H(n))\\ &\quad-(EX_nI(|X_n|<H(n)))^2-(EX_nI(|X_n|<\varepsilon_nH(n))^2)\end{aligned}$$

$$\leqslant \sum_{n=1}^{\infty} \frac{3}{H(n)^2} H(n)^2 P\{ |X_n| \geqslant \varepsilon_n H(n)\} < \infty$$

by (2.6.12). Therefore, by the well-known Kolmogorov strong law of large numbers, we have

$$\sum_{i=1}^{n} (Y_i - \overline{Y}_i) = o(H(n)) \qquad \text{a.s.} \tag{2.6.19}$$

Using (2.6.12) again, we have

$$\sum_{i=1}^{n} EX_i I(|X_i| \leqslant H(i)) - \sum_{i=1}^{n} EX_i I(|X_i| \leqslant \varepsilon_i H(i)) = o(H(n)) \qquad \text{a.s.} \tag{2.6.20}$$

and

$$P(X_n \neq X_n I(|X_n| \leqslant \varepsilon_n H(n)), \text{i.o.}) = 0. \tag{2.6.21}$$

From (2.6.18)—(2.6.21), we finally conclude that

$$\sum_{i=1}^{n} (X_i - EX_i I(|X_i| \leqslant H(i))) - \sum_{i=1}^{n} Y_i = o(H(n)) \qquad \text{a.s.} \tag{2.6.22}$$

as desired. This completes the proof of Theorem 2.6.4.

From Theorem 2.6.4, it is easy to see that

**Theorem 2.6.5** *Let* $\{H(n);\ n \geqslant 1\}$ *be a non-decreasing sequence of positive numbers and* $\{X_n; n \geqslant 1\}$ *be a sequence of independent random variables with* $EX_n = 0$, $EX_n^2 < \infty$. *Assume that there exist* $0 < \delta \leqslant 1$, $\theta \geqslant 0$, $C_1, C_2 > 0$, $\alpha > 0$ *such that* (2.6.8) *and for any* $n \geqslant 1$

$$C_1 n^{\theta} \leqslant E|X_n|^2 \leqslant (E|X_n|^{2+\delta})^{\frac{2}{2+\delta}} \leqslant C_2 n^{\theta}, \tag{2.6.23}$$

$$H(n) \geqslant C_1 n^{\theta/2+\alpha} \tag{2.6.24}$$

*are satisfied. Then* (2.6.11) *holds true.*

### 2.6.2 How big are the increments of partial sums?

Before applying Theorems 2.6.4 and 2.6.5 to study the increments of partial sums of independent non-identically distributed random variables, we first restate Theorem 1.1.4 in the following way.

**Theorem 2.6.6** *Let $\{a_N; N\geq 1\}$, $\{b_N; N\geq 1\}$ be sequences of non-negative integers and $\{Y_n; n\geq 1\}$ be a sequence of independent normal random variables with $Y_n \sim N(0, \sigma_n^2)$. Put*

$$\sigma_{n,N}^2=\sum_{i=k+1}^{k+N}\sigma_i^2,\ \beta_{k,n}=\{2\sigma_{k,n}^2(\log\sigma_{o,n+k}^2/\sigma_{k,n}^2+\log\log\sigma_{k,n}^2)\}^{-1/2},$$

$$\alpha_{n,N}=\{2\sigma_{n,a_N}^2(\log(\frac{\sigma_{o,a_N+b_N}^2}{\sigma_{n,a_N}^2})+\log\log\sigma_{o,a_N+b_N}^2)\}^{-1/2}.$$

*Assume that there exists a constant $A>0$ such that for each $N\geq 2$*

$$\sum_{i=b_{N-1}+1}^{b_N}\sigma_i^2\leq A\sum_{i=b_N+1}^{a_N+b_N}\sigma_i^2,\tag{2.6.25}$$

$$\sum_{i=1}^{a_N+b_N}\sigma_i^2\leq A\sum_{i=1}^{a_{N-1}+b_{N-1}}\sigma_i^2,\tag{2.6.26}$$

$$\lim_{N\to\infty}\min_{0\leq n\leq b_N}\sigma_{n,a_N}^2=\infty.\tag{2.6.27}$$

Then

$$\overline{\lim_{N\to\infty}}\max_{0\leq n\leq b_N}\max_{a_N\leq k<\infty}\max_{1\leq j\leq k}\beta_{n,k}|\sum_{i=n+1}^{n+j}Y_i|=1\quad\text{a.s.}\tag{2.6.28}$$

$$\overline{\lim_{N\to\infty}}\alpha_{b_N,N}|\sum_{i=b_N+1}^{b_N+a_N}Y_i|=1\quad\text{a.s.}\tag{2.6.29}$$

Shao (1989) proved the following theorems.

**Theorem 2.6.7** (Shao 1989) *Let $\{a_N; N\geq 1\}$ and $\{b_N; N\geq 1\}$ be sequences of non-negative integers and $\{H(n); n\geq 1\}$ and $\{X_n; n\geq 1\}$ be as in Theorem 2.6.4 satisfying (2.6.8)—(2.6.10). Put*

$$\sigma_{n,k}^{*2}=\sum_{i=n+1}^{n+k}EX_i^2,\ \beta_{n,k}^*=(2\sigma_{n,k}^{*2}(\log\frac{\sigma_{o,n+k}^{*2}}{\sigma_{n,k}^{*2}}+\log\log\sigma_{n,k}^{*2}))^{-1/2}$$

$$\alpha_{n,N}^*=\{2\sigma_{n,a_N}^{*2}(\log\frac{\sigma_{o,a_N+b_N}^{*2}}{\sigma_{n,a_N}^{*2}}+\log\log\sigma_{o,a_N+b_N}^{*2})\}^{-1/2}$$

*Suppose that there exists a constant $A>0$ such that for each $N\geq 2$*

$$\sum_{i=b_{N-1}+1}^{b_N}EX_i^2\leq A\sum_{i=b_N+1}^{a_N+b_N}EX_i^2,\tag{2.6.30}$$

$$\sum_{i=1}^{a_N+b_N} EX_i^2 \leqslant A \sum_{i=1}^{a_{N-1}+b_{N-1}} EX_i^2, \tag{2.6.31}$$

$$H^2(n+a_N) \leqslant A\sigma_{n,a_N}^{*2}\left(\log \frac{\sigma_{o,n+a_N}^{*2}}{\sigma_{n,a_N}^{*2}} + \log\log \sigma_{n,a_N}^{*2}\right) \tag{2.6.32}$$

*for* $0 \leqslant n \leqslant b_N$ *and any* $N \geqslant 1$. *Then, we have*

$$\overline{\lim_{N\to\infty}} \max_{0\leqslant n\leqslant b_N} \max_{1\leqslant j\leqslant a_N} \beta_{n,a_N}^* \left| \sum_{i=n+1}^{n+j} (X_i - EX_i I(|X_i| \leqslant H(i)))\right| = 1 \quad \text{a.s.} \tag{2.6.33}$$

$$\overline{\lim_{N\to\infty}} \alpha_{b_N,N}^* \left| \sum_{i=b_N+1}^{b_N+a_N} (X_i - EX_i I(|X_i| \leqslant H(i)))\right| = 1 \quad \text{a.s.} \tag{2.6.34}$$

*Proof* Put $\sigma_i^2 = \operatorname{Var} X_i I(|X_i| \leqslant H(i))$. By Theorems 2.6.4 and 2.6.6 and (2.6.32), we have

$$\overline{\lim_{N\to\infty}} \max_{0\leqslant n\leqslant b_N} \max_{1\leqslant j\leqslant a_N} \beta_{n,a_N} \left| \sum_{i=n+1}^{n+j} (X_i - EX_i I(|X_i| \leqslant H(i)))\right| = 1 \quad \text{a.s.} \tag{2.6.35}$$

$$\overline{\lim_{N\to\infty}} \alpha_{b_N,N} \left| \sum_{i=b_N+1}^{a_N+b_N} (X_i - EX_i I(|X_i| \leqslant H(i)))\right| = 1 \quad \text{a.s.} \tag{2.6.36}$$

To complete the proof, it suffices to show that

$$\beta_{n,a_N}/\beta_{n,a_N}^* \to 1 \text{ as } N \to \infty \text{ uniformly in } 0 \leqslant n \leqslant b_N \tag{2.6.37}$$

and

$$\alpha_{b_N,N}/\alpha_{b_N,N}^* \to 1 \text{ as } N \to \infty. \tag{2.6.38}$$

It follows from (2.6.9) and (2.6.10) that

$$\begin{aligned} EX_i^2 &\geqslant \operatorname{Var} X_i I(|X_i| \leqslant H(i)) = \sigma_i^2 \\ &\geqslant EX_i^2 - 2EX_i^2 I(|X_i| > H(i)) \\ &\geqslant EX_i^2 - 2E|X_i|^{2+\delta}/H(i)^\delta \\ &\geqslant EX_i^2 - 2CEX_i^2 H(i)^{\delta-\alpha}/H(i)^\delta \\ &= EX_i^2(1 - 2C/H(i)^\alpha) \\ &\geqslant EX_i^2(1 - 2C^{1+\alpha}/i^{\alpha^2}). \end{aligned}$$

From this inequality, we can deduce that (2.6.37) and (2.6.38) hold. This completes the proof of Theorem 2.6.7.

As a corollary of Theorem 2.6.7, we write the following conclusion, whose proof is left as an exercise for the reader.

**Theorem 2.6.8** *Let* $\{H(n); n \geqslant 1\}$ *be a non-decreasing sequence of positive numbers,* $\{a_N; N \geqslant 1\}$ *and* $\{b_N; N \geqslant 1\}$ *be sequences of nonnegative integers, and* $\{X_n; n \geqslant 1\}$ *be a sequence of independent random variables with* $EX_n = 0$ *and* $EX_n^2 < \infty$. *Assume that there exist* $0 < \delta < 1$, $\theta \geqslant 0$, $C_1$, $C_2 > 0$, $\alpha > 0$ *such that* (2.6.8), (2.6.23) *and the following conditions are satisfied:*

$$b_N - b_{N-1} \leqslant A a_N, \tag{2.6.39}$$

$$b_N + a_N \leqslant A(a_{N-1} + b_{N-1}), \tag{2.6.40}$$

$$H^2(n)/n^{\theta+\alpha} \text{ is non-decreasing.} \tag{2.6.41}$$

$$a_N \geqslant C_1 \frac{H^2(b_N + a_N)}{(b_N + a_N)^\theta (\log(\frac{b_N + a_N}{a_N}) + \log\log(b_N + a_N))}. \tag{2.6.42}$$

*Then* (2.6.33) *and* (2.6.34) *hold true.*

Chapter 3

# Strong Laws of the Processes Generated by Infinite Dimensional Ornstein-Uhlenbeck Processes

## 3.1 Introduction

A real-valued stationary Gaussian process $\{X(t), -\infty < t < \infty\}$ will be called an Ornstein-Uhlenbeck (OU) process with coefficients $\gamma$ and $\lambda$($\gamma, \lambda > 0$) if $EX(t) = 0$ and

$$\Gamma(s,t) = EX(t)X(s) = (\gamma/\lambda)\exp(-\lambda|t-s|). \qquad (3.1.1)$$

Let $Y(t) = \{X_1(t), \cdots, X_i(t), \cdots\}$, where $X_i(\cdot)$ are independent OU processes with coefficients $\gamma_i$ and $\lambda_i (i=1,2,\cdots)$. The infinite-dimensional OU process $Y(\cdot)$ has been extensively studied in the literature of the past twenty or so years with several different applications in mind. For example, it was used to describe physical phenomena subject to random forces in Dawson (1972), and appeared in the infinite-dimensional filtering and quantum string theory in Miyahara (1982) and was also suggested as a model for certain biological systems in Dawson (1972) and Walsh (1981). For a more detailed and accurate discussion along these lines we refer to Antoniadis and Carmona (1987).

Dawson (1972) first studied $Y(\cdot)$ as the stationary solution of an infinite array of stochastic differential equations

$$dX_i(t) = -\lambda_i X_i(t)dt + (2\gamma_i)^{1/2} dW_i(t) \quad i = 1, 2, \cdots, \qquad (3.1.2)$$

where $\{W_i(t), -\infty < t < \infty\}$ are independent Wiener processes. If we assume $\Gamma_0 = \sum_{i=1}^{\infty} \gamma_i/\lambda_i < \infty$, then $Y(t)$ is almost surely (a.s.) an $l^2$-valued OU process ($E\|Y(t)\|_{l^2}^2 = \Gamma_0$) at fixed times. In the case of $\gamma_k = 1$ for all $k$, and for

large $i$, we have also $ci^{1+\delta} \leqslant \lambda_i \leqslant di^{1+\delta}$ for some $c>0$, $d>0$ and $\delta>0$, then Dawson (1972) showed that $Y(\cdot)$ in $l^2$ is a. s. continuous. Since the coordinate OU processes $X_k(\cdot)$ are continuous, it follows from standard Hilbert space theory that to demonstrate $l^2$ continuity of $Y(\cdot)$ it is enough to show that the real-valued process $\chi^2(\cdot) = \| Y(\cdot) \|_{l^2}^2$ is continuous. Iscoe and McDonald (1986), Schmuland (1988b) developed techniques for studying the latter process and showed that $\chi^2(\cdot)$, and hence also $Y(\cdot)$ in $l^2$, is continuous if, in addition to $\Gamma_0 < \infty$, we have also the condition $\Gamma_2 = \sum_{i=1}^{\infty} \gamma_i^2/\lambda_i < \infty$ .

This result is not sharp in that $\gamma_k$ can be a lot larger and we will still have continuity. Iscoe, Marcus, McDonald, Talagrand and Zinn (1990) showed for example that if, in addition to the finiteness of $\Gamma_0$, we have also $\max_{k \geqslant 1} \gamma_k ((\log \gamma_k) \vee 0)^r/(\lambda_k \vee 1) < \infty$ for some $r>1$, then $Y(\cdot)$ is a.s. $l^2$ continuous. In a somewhat more general context, Fernique (1989) gave a complete solution for the latter continuity problem. A special case of his theorem reads as follows: For each $x \in R$, let $K(x) = \{ k \in N ; \gamma_k > \lambda_k x \}$ and $\lambda(x) = \sup \{ \lambda_k : k \in K(x) \}$. Then $Y(\cdot) \in l^2$ is a.s. continuous if and only if we have $\Gamma_0 < \infty$ and $\int ([\log(\lambda(x))] \vee 0)\, dx < \infty$ as well. Consequently, (cf. Corollary 1 of Fernique 1989), for $Y(\cdot) \in l^2$ to be a.s. continuous, it is sufficient that we have $\sum_{k=1}^{\infty} (\gamma_k/\lambda_k)(1 + ((\log \lambda_k) \vee 0)) < \infty$.

On the other hand, the finiteness of $\Gamma_2$ gives more than just continuity of $Y(\cdot)$ in $l^2$. Using variations of this condition, Schmuland (1988a) established various orders of the Hölder continuity for $Y(\cdot)$ in $l^2$ as well as for $\chi^2(\cdot)$, while Csörgő and Lin (1990b) obtained exact moduli of continuity for $\chi^2(\cdot)$ under the condition $\Gamma_2 < \infty$. And further Lin (1990c) established a logarithmic-type law for $\chi^2(\cdot)$.

Another real-valued process which is also closely related to $Y(\cdot) \in l^2$ is the stationary mean zero Gaussian process $X(\cdot)$ defined by

$$X(t) = \sum_{k=1}^{\infty} X_k(t), \qquad -\infty < t < \infty , \tag{3.1.3}$$

where $X_k(\cdot)$ are again the coordinate processes of $Y(\cdot)$. This process can of course be studied by well-developed techniques for Gaussian processes. In particular $X(\cdot)$ is a.s. continuous if and only if it satisfies Fernique's necessary and sufficient condition for the continuity of a stationary Gaussian process (cf. Corollary 2.5 of Section IV. 2 in Jain and Marcus 1978), i.e. if and only if in this case $E|X(t)-X(s)|^2=g^2(|t-s|)$, where $g(u)$ is an increasing function in $u>0$, we have that $g(u)/(u(\log(1/u))^{1/2})$ is integrable at zero. Using this condition one can also compare the processes $Y(\cdot)\in l^2$ with $X(\cdot)$. For example, the condition of finiteness of $\Gamma_2$ with $\gamma_k=1$ $(k=1,2,\cdots)$ reduces to that of $\Gamma_0$, and hence it is sharp for the a.s. continuity of $Y(\cdot)$ in $l^2$. However, in this case Iscoe and McDonald (1986, Example 3 (due to D. A. Dawson)) showed that with $\sum_{k=1}^{\infty}\lambda_k^{-1}<\infty$ but $\sum_{k=1}^{\infty}\lambda_k^{-1}(\log\lambda_k)^{1/2}=\infty$ (e.g. $\lambda_k=k(\log k)^{3/2}$), $Y(\cdot)\in l^2$ is a.s. continuous but $X(\cdot)$ does not satisfy the above mentioned Fernique's condition. On the other hand, we have

$$E\|Y(t)-Y(s)\|_{l^2}^2=E|X(t)-X(s)|^2,$$

and consequently, in general, checking Fernique's necessary and sufficient condition for the a.s. continuity of the real-valued, stationary, mean zero Gaussian process $X(\cdot)$ should be also sufficient for that of the stationary, mean zero Gaussian process $Y(\cdot)$ in $l^2$. Indeed, Csáki, Csörgő, Lin and Révész (1990) showed that the Gaussian process $Y(\cdot)$ is continuous with probability one if for some $\delta>0$ we have $\sum_{k=1}^{\infty}\gamma_k(\log(\lambda_k\vee e))^{1+\delta}/\lambda_k<\infty$, and they also established moduli of continuity properties of $X(\cdot)$. The moduli of continuity for the latter process were also proved by Csörgő and Lin (1990b). Just like studying the process $\chi^2(\cdot)$ on its own, that of $X(\cdot)$ is also of interest. For example, when proposing mathematical models for neural response, one of the processes figuring in Walsh's work (1981) is $X(\cdot)$.

Another natural way of studying the sequence of OU processes $\{X_k(\cdot)\}$ of $Y(\cdot)$ is with regard to the path behavior of their partial sum process defined by

$$X(t,n)=\sum_{k=1}^{n} X_k(t), \qquad -\infty<t<\infty, n=1,2,\cdots \tag{3.1.4}$$

as a two-time parameter stochastic process on $R\times Z^+$, where $Z^+$ is the set of all non-negative integers and $X(t,0)\equiv 0$ for all $t\in R$. Then $EX(t,n)=0$ and the covariance function of this process is given by

$$\begin{aligned}\Gamma(m,n,s,t) &:= EX(s,m)X(t,n) \\ &= \sum_{k=1}^{m\wedge n}(\gamma_k/\lambda_k)\exp(-\lambda_k\,|t-s|),\ m\wedge n=1,2,\cdots.\end{aligned} \tag{3.1.5}$$

The case of the OU processes $X_k(\cdot)$ having the same coefficients $\gamma_k=\gamma$, $\lambda_k=\lambda$ for all $k$ means that with $\Gamma(s,t)$ as in (3.1.1), (3.1.5) becomes

$$\Gamma(m,n,s,t)=(m\wedge n)\Gamma(s,t),$$

i.e. $X(t,n)$ is like a Wiener process in $n$ and an OU process in $t$. This is in complete analogy with summing i.i.d. one-time parameter Wiener processes for the sake of producing a two-time parameter Wiener sheet, or like summing i.i.d. Brownian bridges in order to get a Kiefer process (cf. Sections 1.11 and 1.15 in Csörgő and Révész 1981). The stationarity of the Gaussian process $X(t,n)$ in $t$ and the nature of its Wiener process in $n$ prompted Csörgő and Lin (1990a) to study its sample path fluctuation in $t$ and $n$ along the lines of Chapter 1 in Csörgő and Révész (1981), where the a.s. behavior of some increments of the Wiener sheet and the Kiefer process is established. Csörgő and Lin (1988a) also obtained a law of the iterated logarithm for $\{X(t,n)\}$, which was improved by Shao (1990).

Another source of motivation for studying $\{X(t,n)\}$ is provided by the classical Müntz-Szász theorem (cf., e.g., Section 15. 25 in Rudin 1966) which states that the set of all finite linear combinations of the functions

$$\{1, x^{\lambda_1}, x^{\lambda_2}, \cdots\},\ 0<\lambda_1<\lambda_2<\cdots \tag{3.1.6}$$

is dense in $C[0,1]$ with the supremum norm if and only if $\Sigma\lambda_n^{-1}=\infty$. Following Rudin (1966) we will express this by simply saying that the functions in (3. 1. 6) span $C[0,1]$. Let $x=e^t$. Then the Müntz-Szász theorem implies that the functions $\{e^{\lambda_i t}\}$ span $C(-\infty,0]$, and so the

functions

$$\{ e^0, e^{-\lambda_1 t}, e^{-\lambda_2 t}, \cdots \}, \quad 0<\lambda_1<\lambda_2<\cdots \tag{3.1.7}$$

span $C[0,\infty)$ with the supremum norm $\|f-g\|=\sup_{t>0}|f(t)-g(t)|$ if and only if $\Sigma\lambda_n^{-1}=\infty$. Relating the latter to the covariance structure (3.1.5) of the two-time parameter Gaussian process $X(t,n)$ suggests that the path behavior of a large class of stationary Gaussian processes be quite similar to that of $X(t,n)$, provided the coefficients $\{\lambda_n\}$ do not tend to infinity too fast.

In the last section of this chapter, we will introduce a more general two-parameter Gaussian process, which is a generalization of common processes, such as the two-parameter Wiener process, Kiefer process and a continuous time parameter version of the process $X(t,n)$ of (3.1.4). Path propcrtics of this process will be studicd.

## 3.2 Partial Sum Process

At first, we investigate the path properties of the two-parameter Gaussian process $\{X(t,n), -\infty<t<\infty, n=1,2\cdots\}$ defined by (3.1.4), We will establish the moduli of continuity and large increment results. A law of the iterated logarithm is also given. All are based on large deviation inequalities.

### 3.2.1 Large deviations

In order to obtain the increment results and the law of iterated logarithm for $\{X(t,n)\}$, we need the following large deviation inequalities.

The inequalities from now on are considered to hold true for $N$ large. Let

$$\sigma_N^2(h)=2\sum_{i=1}^{N}\frac{\gamma_i}{\lambda_i}(1-e^{-\lambda_i h}), \quad \sigma_N^2=\sigma_N^2(h_N),$$

where $\{h_N]$ is a sequence of positive numbers,

$$\Gamma_{0N}=\sum_{i=1}^{N}\frac{\gamma_i}{\lambda_i}, \Gamma_{1N}=\sum_{i=1}^{N}\gamma_i \quad \text{and} \quad \lambda_N'=\max_{1\leqslant i\leqslant N}\lambda_i.$$

**Lemma 3.2.1** *Suppose that there exists a constant $A>0$ such that*

$$\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i>1/h}} \gamma_i/\lambda_i \leqslant Ah \sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i\leqslant 1/h}} \gamma_i \quad \text{for } 0\leqslant h\leqslant h_N . \tag{3.2.1}$$

*Then $\sigma_N^2(h)/h^\alpha$ is an increasing function of $h$ on $(0,h_N)$ for every $N$, where $\alpha=1/(3(1+A))$.*

*Proof* Let $f(h)=f_N(h)=\sigma_N^2(h)/h^\alpha$. For $0<h<h_N$ we have

$$f'(h)=h^{-\alpha-1}\Big(-\alpha\sum_{i=1}^N \frac{\gamma_i}{\lambda_i}(1-e^{-\lambda_i h})+\sum_{i=1}^N \gamma_i h e^{-\lambda_i h}\Big)$$

$$\geqslant h^{-\alpha-1}\Big(-\alpha\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i\leqslant 1/h}}\gamma_i h-\alpha\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i>1/h}}\frac{\gamma_i}{\lambda_i}+\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i\leqslant 1/h}}\gamma_i h e^{-\lambda_i h}\Big)$$

$$\geqslant h^{-\alpha-1}\Big(-\alpha(1+A)h\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i\leqslant 1/h}}\gamma_i+\frac{1}{3}h\sum_{\substack{1\leqslant i\leqslant N\\ \lambda_i\leqslant 1/h}}\gamma_i\Big)\geqslant 0$$

as desired.

**Lemma 3.2.2** *Let $\{T_N\}$ and $\{h_N\}$ be sequences of positive numbers with $h_N\leqslant T_N$. Suppose that Condition (3.2.1) is satisfied. Then for any $\varepsilon>0$ there exists a constant $C=C(\varepsilon)>0$ such that the inequality*

$$P\{\sup_{|t|\leqslant T_N}\ \sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|\geqslant v\sigma_N\}\leqslant (CT_N/h_N)\exp\Big(-\frac{v^2}{2+\varepsilon}\Big) \tag{3.2.2}$$

*holds for any $v>0$.*

*Proof* By Lemma 3.2.1, $\sigma_N^2(h)/h^\alpha$ is increasing for $0<h\leqslant h_N$, where $\alpha=1/(3(1+A))$. Let $r=r(\varepsilon)$ be a positive number to be specified later on. Putting $r_1=h_N/2^r$ and $t_r=[t/r_1]r_1$, we have

$$\begin{aligned}|X(t+s,N)-X(t,N)|\leqslant{}& |X((t+s)_r,N)-X(t_r,N)|\\ &+\sum_{j=0}^{\infty}|X((t+s)_{r+j+1},N)-X((t+s)_{r+j},N)|\\ &+\sum_{j=0}^{\infty}|X(t_{r+j+1},N)-X(t_{r+j},N)|.\end{aligned} \tag{3.2.3}$$

By choosing $r = r(\varepsilon)$ to be large enough we get

$$P\{\sup_{|t|\leqslant T_N} \sup_{0\leqslant s\leqslant h_N} |X((t+s)_r, N) - X(t_r, N)| \geqslant v(1-\varepsilon/6)\sigma_N\} \tag{3.2.4}$$

$$\leqslant \frac{4T_N h_N}{r_1^2} \exp\left(-\frac{v^2}{2+\varepsilon}\right) \leqslant (CT_N/h_N)\exp\left(-\frac{v^2}{2+\varepsilon}\right),$$

$$P\{\sup_{|t|\leqslant T_N} \sup_{0\leqslant s\leqslant h_N} \sum_{j=0}^{\infty} |X(t+s)_{r+j+1}, N) - X((t+s)_{r+j}, N) \tag{3.2.5}$$

$$\leqslant \sum_{j=0}^{\infty} \sigma_N (v^2 + 6j)/2^{\alpha(r+j+1)})^{1/2}\}$$

$$\geqslant \sum_{j=0}^{\infty} P(\sup_{|t|\leqslant T_N} \sup_{0\leqslant s\leqslant h_N} |X((t+s)_{r+j+1}, N) - X((t+s)_{r+j}, N)$$

$$\geqslant \sigma_N((v^2+6j)/2^{\alpha(r+j+1)})^{1/2}\}$$

$$\leqslant \sum_{j=0}^{\infty} (4T_N/h_N)2^{2(r+j+1)}\exp\left(-\frac{v^2+6j}{2+\varepsilon}\right)$$

$$\leqslant (CT_N/h_N)\exp\left(-\frac{v^2}{2+\varepsilon}\right)$$

and similarly

$$P\{\sup_{|t|\leqslant T_N} \sup_{0\leqslant s\leqslant h_N} \sum_{j=0}^{\infty} |X(t_{r+j+1}, N) - X(t_{r+j}, N)| \tag{3.2.6}$$

$$\geqslant \sum_{j=0}^{\infty} \sigma_N((v^2+6j)/2^{\alpha(r+j+1)})^{1/2}\}$$

$$\leqslant (CT_N/h_N)\exp\left(-\frac{v^2}{2+\varepsilon}\right).$$

We can assume without loss of generality that $v \geqslant 1$. Then

$$\sum_{j=0}^{\infty}\left(\frac{v^2+6j}{2^{\alpha(r+j-1)}}\right)^{1/2} \leqslant \frac{v}{2^{\alpha r/2}} \sum_{j=0}^{\infty} \frac{1}{2^{\alpha(j+1)/2}} + \frac{1}{2^{\alpha r/2}} \sum_{j=0}^{\infty}\left(\frac{6j}{2^{\alpha(j+1)}}\right)^{1/2} \leqslant \frac{\varepsilon}{12} v,$$

provided that $r = r(\varepsilon)$ is large enough.

Now the proof of (3.2.2) is completed by combining these inequalities.

*Remark* 3.2.1 If $h_N \leqslant \lambda'^{-1}_N$, Condition (3.2.1) is satisfied automatically.

Furthermore, we have

**Lemma 3.2.3** *Corresponding to* (3.2.2), *we have*

$$P\{\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,n)-X(t,n)|\geqslant v\sigma_N\} \tag{3.2.7}$$
$$\leqslant (C'\sigma_N/h_N)\exp(-\frac{v^2}{2+\varepsilon})$$

*for some* $C'=C'(\varepsilon)>0$, *provided* $v\geqslant(3(\log 2C))^{1/2}/\varepsilon$, *where* $C=C(\varepsilon)$ *is defined in* (3.2.2).

*Proof* Define

$$E_1=\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,1)-X(t,1)|\geqslant v\sigma_N\},$$
$$E_i=\{\max_{l<i}\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,l)-X(t,l)|<v\sigma_N$$
$$\leqslant\sup_{|t|<h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,i)-X(t,i)|\},\ i=2,\cdots,N.$$

We have for $0<\varepsilon<1$

$$A_N:=\{\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,n)-X(t,n)|\geqslant v\sigma_N\}=\bigcup_{n=1}^{N}E_n$$
$$\subset\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|\geqslant(1-\varepsilon)v\sigma_N\}$$
$$\bigcup(\bigcup_{n=1}^{N-1}(E_N\bigcap\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|<(1-\varepsilon)v\sigma_N\}))$$
$$\subset\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|\geqslant(1-\varepsilon)v\sigma_N\}$$
$$\bigcup(\bigcup_{n=1}^{N-1}(E_N\bigcap\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|(X(t+s,N)-X(t,N))-(X(t+s,n)$$
$$-X(t,n))|>\varepsilon v\sigma_N\})).$$

Noting that $\{(X(t+s,N)-X(t,N))-(X(t+s,n)-X(t,n)),|t|<h_N,$ $0\leqslant s\leqslant h_N\}$ and $E_n$ are independent and using (3.2.2), we obtain

$$P(A_N)\leqslant P\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)|\geqslant(1-\varepsilon)v\sigma_N\}$$
$$+\sum_{n=1}^{N-1}P(E_n)P\{\sup_{|t|\leqslant h_N}\sup_{0\leqslant s\leqslant h_N}|X(t+s,N)-X(t,N)$$

$$- (X(t+s,n) - X(t,n))| > \varepsilon v \sigma_N \}$$

$$\leqslant P\{ \sup_{|t| \leqslant h_N} \sup_{0 \leqslant s \leqslant h_N} |X(t+s,N) - X(t,N)| \geqslant (1-\varepsilon) v \sigma_N \}$$

$$+ C \sum_{n=1}^{N-1} P(E_n) \exp\{ -\varepsilon^2 v^2 \sigma_N^2 / (2+\varepsilon) \sum_{i=n+1}^{N} \frac{\gamma_i}{\lambda_i} (1 - e^{-\lambda_i h_N}) \}$$

$$\leqslant P\{ \sup_{|t| \leqslant h_N} \sup_{0 \leqslant s \leqslant h_N} |X(t+s,N) - X(t,N)| \geqslant (1-\varepsilon) v \sigma_N \}$$

$$+ P(A_N)/2 .$$

provided $v^2 \geqslant 3(\log 2C)/\varepsilon^2$. This inequality yields

$$P(A_N) \leqslant 2P\{ \sup_{|t| \leqslant h_N} \sup_{0 \leqslant s \leqslant h_N} |X(t+s,N) - X(t,N)| \geqslant (1-\varepsilon) v \sigma_N \}$$

which implies the desired inequality immediately.

**Lemma 3. 2. 4** *Let $\{T_N\}$, $\{h_N\}$, $\{h'_N\}$ and $\{h''_N\}$ be sequences of positive numbers with $h_N \leqslant T_N$ and $h'_N \leqslant h_N \leqslant h''_N$. Suppose that $\{h_N\}$ is non-increasing and Condition* (3.2.1) *is satisfied. Then for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that the inequality*

$$P\{ \max_{1 \leqslant n \leqslant N} \sup_{|t| \leqslant T_N} \sup_{0 \leqslant s \leqslant h''_N} |X(t+s,n) - X(t,n)|$$

$$/(\sigma_{nN}(s)(v^2 + (2+\varepsilon)\log\log(\sigma_{nN}^2(s) + \sigma_{nN}^{-2}(s)))^{1/2} + \sigma_{nN}(s)) \geqslant 1\}$$

$$\leqslant (C T_N h''_N / h'^2_N) \exp\left(-\frac{v^2}{2+\varepsilon}\right) \qquad (3.2.8)$$

holds for any $v > 0$, where $\sigma_{nN}(s) = \sigma_n(s) + \varepsilon \sigma_n(h'_N)$.

*Proof* Let $r = r(\varepsilon)$ be a positive number specified later on. Put $r_1 = h'_N / 2^{2^r}$ and $t_r = [t/r_1] r_1$. We can also write inequality (3. 2. 3). By Lemma 3.2.1 (noting that $h'_N \leqslant h_n$ for each $n < N$ since $\{h_N\}$ is non-increasing), it is easy to see that

$$(E(X((t+s)_r, n) - X(t_r, n))^2)^{1/2}$$

$$\leqslant (E(X((t+s)_r, n) - X(t,n))^2)^{1/2} + (E(X(t,n) - X(t_r, n))^2)^{1/2}$$

$$\leqslant \sigma_n(s) + \sigma_n(r_1) \leqslant \sigma_{nN}(s)$$

provided that $r$ is large enough. Fix $|t| \leqslant T_N$ and $0 \leqslant s \leqslant h''_N$. For any $0 < \varepsilon < 1/2$, let $\theta = 1 + \varepsilon/16$ and $A_k = \{ n : n \leqslant N, \theta^k \leqslant \sigma_{nN}(s) < \theta^{k+1} \}$. Put

$$u_n(s) = \sigma_{nN}(s)(v^2 + (2+\varepsilon)\log\log(\sigma_{nN}^2(s) + \sigma_{nN}^{-2}(s)))^{1/2} + \sigma_{nN}(s),$$

$$u(k) = \theta^k(v^2 + (2+\varepsilon)\log\log\theta^{|2k|})^{1/2}.$$

Then, noting that $\{ X((t+s)_r, n) - X(t_r, n),\ n \geqslant 1 \}$ is a sequence with independent increments, we have

$$\begin{aligned} & P\{ \max_{1 \leqslant n \leqslant N} |X((t+s)_r, n) - X(t_r, n)|/u_n(s) \geqslant 1 - \varepsilon/8 \} \\ & \leqslant \sum_k P\{ \max_{n \in A_k} |X((t+s)_r, n) - X(t_r, n)| \geqslant (1 - \varepsilon/8) u(k) \} \\ & \leqslant \sum_k \exp\{ -\frac{1}{2}\theta^{-2}(1 - \frac{\varepsilon}{8})^2 (v^2 + (2+\varepsilon)\log\log\theta^{|2k|}) \} \\ & \leqslant c \exp(-\frac{v^2}{2+\varepsilon}). \end{aligned} \tag{3.2.9}$$

It follows that

$$\begin{aligned} & P\{ \max_{1 \leqslant n \leqslant N} \sup_{|t| \leqslant T_N} \sup_{0 \leqslant s \leqslant h''_N} |X((t+s)_r, n) - X(t_r, n)|/u_n(s) \geqslant 1 - \varepsilon/8 \} \\ & \leqslant (c T_N h''_N / h'^2_N) \exp(-\frac{v^2}{2+\varepsilon}). \end{aligned}$$

We now deal with the first series of the right-hand side of (3.2.3). By Lemma 3.2.1 again, we have

$$\begin{aligned} \sigma_{ni}^2 &:= E(X((t+s)_{r+j+1}, n) - X((t+s)_{r+j}, n))^2 \\ & \leqslant \sigma_n^2(h'_n / 2^{2^{r+j}}) \leqslant \delta(\varepsilon) 2^{-\alpha 2^j} \sigma_n^2(h'_N) \\ & \leqslant \delta(\varepsilon) 2^{-\alpha 2^j} \varepsilon^{-2} \theta^{2k+2}, \end{aligned}$$

where $\delta(\varepsilon) \leqslant 2^{-\alpha(2^r - 1)}$. Taking $\delta(\varepsilon)$ to be small enough such that

$$\begin{aligned} & \sum_{j=0}^{\infty} (\delta(\varepsilon) 2^{-\alpha 2^j})^{1/2} \varepsilon^{-1} \theta^{k+1} (v^2 + (2+\varepsilon)\log\log(\sigma_n^2(s) + \sigma_n^{-2}(s)) + 2^{r+j+2})^{1/2} \\ & \leqslant \frac{\varepsilon}{16} u_n(s). \end{aligned}$$

It is possible provided $r$ is large enough. Then, similarly to (3.2.9), it follows that

$$P\{\max_{1\leqslant n\leqslant N}\sum_{j=0}^{\infty}|X((t+s)_{r+j+1},n)-X((t+s)_{r+j},n)|/u_n(s)\geqslant \varepsilon/16\}$$

$$\leqslant \sum_{k}\sum_{j=0}^{\infty}P\{\max_{n\in Ak}|X((t+s)_{r+j+1},n)-X((t+s)_{r+j},n)|$$

$$\geqslant(\delta(\varepsilon)2^{-\alpha 2^j})^{1/2}\varepsilon^{-1}\theta^{k+1}(v^2+(2+\varepsilon)\log\log\theta^{|2k|}+2^{r+j+2})^{1/2}\}$$

$$\leqslant \sum_{j=0}^{\infty}\sum_{k}\exp\{-\frac{1}{2}(v^2+(2+\varepsilon)\log\log\theta^{|2k|}+2^{r+j+2})\}$$

$$\leqslant c\sum_{j=0}^{\infty}\exp\{-(\frac{1}{2}v^2+2^{r+j+1})\}.$$

It follows that

$$P\{\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant T_N}\sup_{0\leqslant s\leqslant h_N''}\sum_{j=0}^{\infty}|X((t+s)_{r+j+1},n)-X((t+s)_{r+j},n)|$$

$$/u_n(s)\geqslant\varepsilon/16\}$$

$$\leqslant\sum_{j=0}^{\infty}(c2^{2^{r+j+1}}T_N h_N''/h_N'^2)\exp\{-(\frac{1}{2}v^2+2^{r+j+1})\}$$

$$\leqslant(cT_N h_N''/h_N'^2)\exp(-v^2/2).$$

For the second series of the right-hand side of (3.2.3) we have a similar inequality. Combining these inequalities yields the desired (3.2.8).

**Lemma 3.2.5** *For any $\varepsilon>0$, there exist constants $C=C(\varepsilon)>0$ and $v(\varepsilon)>0$ such that the inequality*

$$P\{\max_{1\leqslant n\leqslant N}\sup_{|t|\leqslant TN}|X(t,n)|\geqslant v\Gamma_{oN}^{1/2}\}\leqslant C(1+T_N\Gamma_{1N}/\Gamma_{oN})\exp(-\frac{v^2}{2+\varepsilon})\tag{3.2.10}$$

*holds provided $v\geqslant v(\varepsilon)$.*

*Proof* We prove that

$$P\{\sup_{|t|<TN}|X(t,N)|\geqslant v\Gamma_{oN}^{1/2}\}\leqslant C(1+T_N\Gamma_{1N}/\Gamma_{oN})\exp(-\frac{v^2}{2+\varepsilon})\tag{3.2.11}$$

for any $v>0$, which implies (3.2.10) by a proof similar to that of Lemma 3.2.3.

Put $d=\delta^2\Gamma_{oN}/\Gamma_{1N}$ where $\delta=\varepsilon/64$. Then

$$P\{\sup_{|t|\leqslant T_N}|X(t,N)|\geqslant v\Gamma_{oN}^{1/2}\} \tag{3.2.12}$$

$$\leqslant 2(1+T_N/d)P\{\sup_{0\leqslant t\leqslant d}|X(t,N)|\geqslant v\Gamma_{oN}^{1/2}\}$$

$$\leqslant 2(1+T_N/d)P\{\sup_{0\leqslant t\leqslant 1}|X(dt,N)|\geqslant v\Gamma_{oN}^{1/2}\}.$$

Note that

$$E(X(dt,N)-X(ds,N))^2=2\sum_{i=1}^{N}\frac{\gamma_i}{\lambda_i}(1-e^{-\lambda_i d|t-s|})$$

$$\leqslant 2\Gamma_{1N}d|t-s|=:\Lambda^2(|t-s|).$$

We have

$$\int_1^\infty \Lambda(e^{-y^2})dy=\int_1^\infty(2\Gamma_{1N}de^{-y^2})^{1/2}dy\leqslant 2\delta\Gamma_{oN}^{1/2}.$$

Using Fernique's Lemma (Lemma 1.5.1) we obtain

$$P\{\sup_{0\leqslant t\leqslant 1}|X(dt,N)|\geqslant v\Gamma_{oN}^{1/2}\}$$

$$\leqslant P\{\sup_{0\leqslant t\leqslant 1}|X(dt,N)|\geqslant\frac{v}{1+8\delta}(\Gamma_{oN}^{1/2}+4\int_1^\infty\Lambda(e^{-y^2})dy)\}$$

$$\leqslant c\int_{v/(1+8\delta)}^\infty e^{-y^2/2}dy\leqslant c\exp(-\frac{v^2}{2+\varepsilon}).$$

Substituting this into (3.2.12) yields (3.2.11). The lemma is proved.

Define $\theta_n(\varepsilon)$ to be the solution of the equation

$$\sum_{i=1}^{N}(\gamma_i/\lambda_i)e^{-2\lambda_i\theta_N(\varepsilon)}=\varepsilon\Gamma_{oN}. \tag{3.2.13}$$

**Lemma 3.2.6** *For any $0<\varepsilon<1/2$, there exists a constant $C=C(\varepsilon)>0$ such that*

$$P\{\sup_{0\leqslant t\leqslant T_N}|X(t,N)|\geqslant v\Gamma_{oN}^{1/2}\} \tag{3.2.14}$$

$$\geqslant C(1+T_N/\theta_N(\varepsilon))\exp(-\frac{v^2}{2(1-2\varepsilon)})$$

*provided $v\geqslant 2(\log(C(\varepsilon)(1+T_N/\theta_N(\varepsilon))))^{1/2}$.*

*Proof* Let $\{ W_i(t),\ t \geqslant 0 \}_{i=1}^{\infty}$ be a sequence of independent standard Wiener processes. Noting that $\{ X(t,N),\ 0 \leqslant t \leqslant T_N \}$ and $\{ \sum_{i=1}^{N} (\gamma_i/\lambda_i)^{1/2} W_i(e^{2\lambda_i t})/e^{\lambda_i t}, 0 \leqslant t \leqslant T_N \}$ have the same distribution, we have

$$P\{ \sup_{0 \leqslant t \leqslant T_N} |X(t,N)| \geqslant v\Gamma_{oN}^{1/2} \} \tag{3.2.15}$$

$$\geqslant P\{ \max_{0 \leqslant j \leqslant T_N/\theta_N} |X(j\theta_N, N)| \geqslant v\Gamma_{oN}^{1/2} \}$$

$$= P\{ \max_{0 \leqslant j \leqslant T_N/\theta_N} | \sum_{i=1}^{N} (\frac{\gamma_i}{\lambda_i})^{1/2} W_i(e^{2\lambda_i j\theta_N})/e^{\lambda_i j\theta_N}| \geqslant v\Gamma_{oN}^{1/2} \}.$$

Put $U_j = \sum_{i=1}^{N} (\frac{\gamma_i}{\lambda_i})^{1/2} W_i(e^{2\lambda_i j\theta_N})/e^{\lambda_i j\theta_N}$, $V_j = \sum_{i=1}^{N} (\frac{\gamma_i}{\lambda_i})^{1/2} W_i(e^{2\lambda_i (j-1)\theta_N}) /e^{\lambda_i j\theta_N}$. It is easy to see that

$$U_j - V_j \sim N(0, \sum_{i=1}^{N} \frac{\gamma_i}{\lambda_i} (1 - e^{-2\lambda_i\theta_N})) = N(0, (1-\varepsilon)\Gamma_{oN})$$

by the definition of $\theta_N$. Whence we have

$$P\{ \max_{0 \leqslant j \leqslant T_N/\theta_N} |U_j| < v\Gamma_{oN}^{1/2} \}$$

$$= P\{ \max_{0 \leqslant j < [T_N/\theta_N]} |U_j| < v\Gamma_{oN}^{1/2}, |U_{[T_N/\theta_N]} - V_{[T_N/\theta_N]} + V_{[T_N/\theta_N]}| < v\Gamma_{oN}^{1/2} \}$$

$$= \int_{-\infty}^{\infty} P\{ |U_{[T_N/\theta_N]} - V_{[T_N/\theta_N]} + y| < v\Gamma_{oN}^{1/2} \} dP\{ V_{[T_N/\theta_N]} < y, \max_{0 \leqslant j < [T_N/\theta_N]} |U_j| < v\Gamma_{oN}^{1/2} \}$$

$$= \int_{-\infty}^{\infty} \{ \Phi(\frac{v\Gamma_{oN}^{1/2} - y}{(1-\varepsilon)^{1/2}\Gamma_{oN}^{1/2}}) - \Phi(\frac{-v\Gamma_{oN}^{1/2} - y}{(1-\varepsilon)^{1/2}\Gamma_{oN}^{1/2}}) \} dP\{ V_{[T_N/\theta_N]} < y, \max_{0 \leqslant j < [T_N/\theta_N]} |U_j| < v\Gamma_{oN}^{1/2} \}$$

$$\leqslant \int_{-\infty}^{\infty} \{ \Phi(\frac{v}{(1-\varepsilon)^{1/2}}) - \Phi(\frac{-v}{(1-\varepsilon)^{1/2}}) \} dP\{ V_{[T_N/\theta_N]} < y, \max_{0 \leqslant j < [T_N/\theta_N]} |U_j| < v\Gamma_{oN}^{1/2} \}$$

$$=\{1-\frac{2}{\sqrt{2\pi}}\int_{v/(1-\varepsilon)^{1/2}}^{\infty}e^{-t^2/2}dt\}P\{\max_{0\leqslant j<[T_N/\theta_N]}|U_j|<v\Gamma_{oN}^{1/2}\}$$

$$\leqslant(1-2Ce^{-v^2/2(1-2\varepsilon)})P\{\max_{0\leqslant j<[T_N/\theta_N]}|U_j|<v\Gamma_{oN}^{1/2}\},$$

here we have used the following facts on the Wiener process :

(a) $U_{[T_N/\theta_N]}-V_{[T_N/\theta_N]}$ and $\{V_{[T_N/\theta_N]}, U_j, 0\leqslant j<[T_N/\theta_N]\}$ are independent.

(b) $\Phi(x-y)-\Phi(-x-y)\leqslant\Phi(x)-\Phi(-x)$ for any $y$ and $x\geqslant 0$.

(c) For any $\delta>0$, there exists a $C(\delta)>0$ such that

$$\int_x^{\infty}e^{-t^2/2}dt\geqslant 2C(\delta)e^{-(1+\delta)x^2/2}\text{ for any }x\geqslant 0.$$

By recurrence, we obtain

$$P\{\max_{0\leqslant j\leqslant T_N/\theta_N}|U_j|<v\Gamma_{oN}^{1/2}\}\leqslant\{1-2C(\varepsilon)e^{-v^2/2(1-2\varepsilon)}\}^{[T_N/\theta_N]+1}\tag{3.2.16}$$

$$\leqslant 1-C(\varepsilon)(1+\frac{T_N}{\theta_N})e^{-v^2/2(1-2\varepsilon)}$$

provided $v\geqslant 2(\log(C(\varepsilon)(1+T_N/\theta_N)))^{1/2}$.

(3.2.14) now follows from (3.2.15) and (3.2.16).

### 3.2.2 Increment results

We will establish some increment results using the large deviation inequalities shown a above. They will be proved both for small $h_N$ and for large $h_N$.

**Theorem 3.2.1** (Csörgő, Lin 1990a) *Let $\{T_N\}$ and $\{h_N\}$ be sequences of positive numbers. Suppose that $\{T_N\}$ is non-decreasing and $\{h_N\}$ is monotonic. And suppose that Condition* (3.2.1) *is satisfied and*

$$\lim_{N\to\infty}(\log(T_N/h_N))/\log\log N=\infty.\tag{3.2.17}$$

*Then we have*

$$\lim_{N\to\infty} \sup_{|t|\leqslant T_N} \alpha_N |X(t+h_N, N) - X(t,N)| = 1 \quad \text{a.s.}$$

$$\lim_{N\to\infty} \max_{1\leqslant n\leqslant N} \sup_{|t|<T_N} \sup_{0\leqslant s\leqslant h_N} \alpha_N |X(t+s,n) - X(t,n)| = 1 \quad \text{a.s.}$$

where

$$\alpha_N = \{ 2\sigma_N^2 (\log T_N/h_N + \log\log(\sigma_N^2 + \sigma_N^{-2}))\}^{-1/2}.$$

*Remark* 3.2.2 When $h_N \to 0$ as $N \to \infty$, this theorem can be viewed as an analogue of the Lévy modulus of continuity of a Wiener process (cf., e.g., Theorems 1.1.1 and 1.14.2 in Csörgő and Révész 1981). When $h_N \to \infty$ as $N \to \infty$, this theorem can be viewed as an analogue of the large increment results of a Wiener process (cf. Theorem 1.2.1 in Csörgő and Révész 1981).

Proof of Theorem 3.2.1.

At first, we prove that for $0<\varepsilon<1/2$

$$\overline{\lim_{N\to\infty}} \max_{1\leqslant n\leqslant N} \sup_{|t|\leqslant T_N} \sup_{0\leqslant s\leqslant h_N} \alpha_N |X(t+s,n) - X(t,n)| \leqslant 1+\varepsilon \quad \text{a.s.} \tag{3.2.18}$$

We first consider the case when $\{h_N\}$ is non-decreasing. At this time it is clear that we can let $\alpha_N = \{2\sigma_N^2[\log T_N/h_N + \log\log \sigma_N^2]\}^{-1/2}$. instead of $\{2\sigma_N^2[\log T_N/h_N + \log\log(\sigma_N^2 + \sigma_N^{-2})]\}^{-1/2}$. Put $\theta = 1+\varepsilon/2$. Define

$$H_{kj} = \{ N : \theta^k < T_N/h_N \leqslant \theta^{k+1}, \theta^j < \sigma_N^2 \leqslant \theta^{j+1} \}, \quad M_{kj} = \max\{ N : N \in H_{kj}\},$$

$$\mathscr{A} = \{ (k,j) : = H_{kj} \neq \emptyset \}, \quad \sigma_{kj}'^2 = \min\{\sigma_N^2 : N \in H_{kj}\}.$$

By the definition

$$1 \leqslant \sigma_{M_{kj}}^2 / \sigma_{kj}'^2 \leqslant \theta.$$

Then we have

$$\overline{\lim_{N\to\infty}} \max_{1\leqslant n\leqslant N} \sup_{|t|\leqslant T_N} \sup_{0\leqslant s\leqslant h_N} \alpha_N |X(t+s,n) - X(t,n)| \tag{3.2.19}$$

$$\leqslant \overline{\lim_{\substack{k\to\infty\\ j\to\infty}}}_{(k,j)\in\mathscr{A}} \max_{N\in H_{kj}} \max_{1\leqslant n\leqslant N} \sup_{|t|\leqslant T_N} \sup_{0\leqslant s\leqslant h_N} \alpha_N |X(t+s,n) - X(t,n)|$$

$$\leqslant \overline{\lim_{\substack{k\to\infty\\ j\to\infty}}}_{(k,j)\in\mathscr{A}} \max_{1\leqslant n\leqslant M_{kj}} \sup_{|t|\leqslant T_{M_{kj}}} \sup_{0\leqslant s\leqslant h_{M_{kj}}} |X(t+s,n) - X(t,n)|$$

$$/\{ 2\sigma_{kj}'^2 \log (\theta^k \log \theta^j )\}^{1/2}$$

$$\leqslant \varlimsup_{\substack{k \to \infty \\ j \to \infty}} \max_{(k,j)\in \mathscr{A}} \sup_{1 \leqslant n \leqslant M_{kj}} \sup_{|t| \leqslant T_{M_{kj}}} \sup_{0 \leqslant s \leqslant h_{M_{kj}}} |X(t+s,n)-X(t,n)|$$

$$/\{ 2\theta^{-1}\sigma^2_{M_{kj}} \log [(\theta^{-1}T_{M_{kj}}/h_{M_{kj}})\log \theta^j ]\}^{1/2}.$$

Using Lemma 3.2.3 we obtain that

$$P\{ \max_{1 \leqslant n \leqslant M_{kj}} \sup_{|t| \leqslant T_{M_{kj}}} \sup_{0 \leqslant s \leqslant h_{M_{kj}}} |X(t+s,n)-X(t,n)|$$

$$/\{ 2\theta^{-1}\sigma^2_{M_{kj}} \log [(\theta^{-1}T_{M_{kj}}/h_{M_{kj}})\log \theta^j ]\}^{1/2} \geqslant 1+\varepsilon \}$$

$$\leqslant (CT_{M_{kj}}/h_{M_{kj}})\exp \{ -\frac{2\theta^{-1}(1+\varepsilon)^2}{2+\varepsilon} \log [(\theta^{-1}T_{M_{kj}}/h_{M_{kj}})\log \theta^j ]\}$$

$$\leqslant C(T_{M_{kj}}/h_{M_{kj}})^{-\varepsilon/2}(j\log \theta)^{-1-\varepsilon/2}$$

$$\leqslant C\theta^{-k\varepsilon/2}(j\log\theta)^{-1-\varepsilon/2}.$$

Thus, using the Borel-Cantelli Lemma, whose generalization in the case of two indicates is trivial, we get (3.2.18) via (3.2.19).

Next, we assume that $\{ h_N \}$ is non-increasing. Define

$$H_k = \{ N : \theta^k < T_N/h_N \leqslant \theta^{k+1} \}, \quad M_k = \max \{ N : N \in H_k \},$$

$$m_k = \min \{ N : N \in H_k \}, \qquad \mathscr{A} = \{ k : H_k \neq \emptyset \}.$$

It is clear that

$$1 \leqslant h_{m_k}/h_{M_k} \leqslant (h_{m_k}/T_{m_k})/(h_{M_k}/T_{M_k}) \leqslant \theta .$$

Then we have

$$\varlimsup_{N \to \infty} \max_{1 \leqslant n \leqslant N} \sup_{|t| \leqslant T_N} \sup_{0 \leqslant s \leqslant h_N} \alpha_N |X(t+s,n)-X(t,n)| \tag{3.2.20}$$

$$\leqslant \varlimsup_{N \to \infty} \max_{1 \leqslant n \leqslant N} \sup_{|t| \leqslant T_N} \sup_{0 \leqslant s \leqslant h_N} (1+3\varepsilon)|X(t+s,n)-X(t,n)|$$

$$/\{ [(\sigma_n(s)+\varepsilon\sigma_n(h_N))(2\log T_N/h_N+(2+\varepsilon)\log \log ((\sigma_n(s)+\varepsilon\sigma_n(h_N))^2$$

$$+(\sigma_n(s)+\varepsilon\sigma_n(h_N))^{-2}))]^{1/2}+\sigma_n(s)+\varepsilon\sigma_n(h_N) \}$$

$$\leqslant \varlimsup_{k \to \infty, k \in \mathscr{A}} \max_{1 \leqslant n \leqslant M_k} \sup_{|t| \leqslant T_{M_k}} \sup_{0 \leqslant s \leqslant h_{m_k}} (1+3\varepsilon)|X(t+s,n)-X(t,n)|$$

$$/\{[\sigma_n(s)+\varepsilon\sigma_n(h_{M_k}))(2\log\theta^k+(2+\varepsilon)\log\log((\sigma_n(s)+\varepsilon\sigma_n(h_{M_k}))^2$$
$$+(\sigma_n(s)+\varepsilon\sigma_n(h_{Mk}))^{-2}))]^{1/2}+\sigma_n(s)+\varepsilon\sigma_n(h_{Mk})\}.$$

Using Lemma 3.2.4, we obtain that

$$P\{\max_{1\leqslant n\leqslant M_k}\sup_{|t|\leqslant TM_k}\sup_{0\leqslant s\leqslant h_{m_k}}|X(t+s,n)-X(t,n)|$$
$$/\{[(\sigma_n(s)+\varepsilon\sigma_n(h_{M_k}))(2\log\theta^k+(2+\varepsilon)\log\log((\sigma_n(s)+\varepsilon\sigma_n(h_{M_k}))^2$$
$$+(\sigma_n(s)+\varepsilon\sigma_n(h_{M_k}))^{-2}))]^{1/2}+\sigma_n(s)+\varepsilon\sigma_n(h_{M_k})\}\geqslant 1+\varepsilon\}$$
$$\leqslant(CT_{M_k}h_{m_k}/h^2_{M_k})\exp\{-\frac{2(1+\varepsilon)^2}{2+\varepsilon}\log\theta^k\}\leqslant C\theta^{-\varepsilon k}$$

which implies (3.2.18) via (3.2.20).

Finally, we prove

$$\underline{\lim}_{N\to\infty}\sup_{|t|\leqslant T_N}\alpha_N|X(t+h_N,N)-X(t,N)|\geqslant 1-\varepsilon\quad\text{a.s.}\tag{3.2.21}$$

For integers $i$ and $j$, $i<j$.

$$E(X((i+1)h_N,N)-X(ih_N,N))(X((j+1)h_N,N)-X(jh_N,N))$$
$$=\sum_{l=1}^{N}E(X_l((i+1)h_N)-X_l(ih_N))(X_l((j+1)h_N)-X_l(jh_N))$$
$$=\sum_{l=1}^{N}\frac{\gamma_l}{\lambda_l}e^{-\lambda_l jh_N}(2e^{\lambda_l ih_N}-e^{\lambda_l(i-1)h_N}-e^{\lambda_l(i+1)h_N})<0.$$

Thus, using Slepian's Lemma (Lemma 1.1.1) and the stationarity of the process, we obtain

$$P\{\max_{|k|\leqslant T_N/h_N}\alpha_N|X((k+1)h_N,N)-X(kh_N,N)|\leqslant 1-\varepsilon\}\tag{3.2.22}$$
$$\leqslant(P\{\alpha_N|X(h_N,N)|\leqslant 1-\varepsilon\})^{2[T_N/h_N]}$$
$$\leqslant\{1-(\exp(-(1-\varepsilon)\log T_N/h_N))/6(\log T_N/h_N)^{1/2}\}^{2[T_N/h_N]}$$
$$\leqslant\exp\{-2(T_N/h_N)^{\varepsilon/2}\}\leqslant N^{-2},$$

where the last inequality is due to Condition (3.2.17). Hence (3.2.21) is proved. Now the two conclusions of Theorem 3.2.1 follow from the combination of (3.2.18) and (3.2.21).

### 3.2.3 The law of the iterated logarithm

**Theorem 3.2.2** (Shao 1990) *Let $\{T_N\}$ be a non-decreasing sequence of positive numbers. Suppose that*

(i) $\Gamma_{0N} \to \infty$ *as* $N \to \infty$,

(ii) *there exists a constant* $d \geqslant 1$ *such that* $\Gamma_{0,N+1} \leqslant d\Gamma_{0N}$ *for every* $N \geqslant 1$,

(iii) $\log(T_N\Gamma_{1N}/\Gamma_{0N}) = o(\log\log\Gamma_{0N})$ *as* $N \to \infty$.

*Then we have*

$$\varlimsup_{N\to\infty} \sup_{|t|\leqslant T_N} |X(t,N)|/(2\Gamma_{0N}\log\log\Gamma_{0N})^{1/2} = 1 \quad \text{a.s.}$$

$$\varlimsup_{N\to\infty} \max_{1\leqslant n\leqslant N} \sup_{|t|\leqslant T_N} |X(t,n)|/(2\Gamma_{0N}\log\log\Gamma_{0N})^{1/2} = 1 \quad \text{a.s.}$$

*Proof* Imitating the proof of (3.2.18), we can also prove that

$$\varlimsup_{N\to\infty} \max_{1\leqslant n\leqslant N} \sup_{|t|\leqslant T_N} |X(t,n)|/(2\Gamma_{0N}\log\log\Gamma_{0N})^{1/2} \leqslant 1 \quad \text{a.s.} \tag{3.2.23}$$

by using Lemma 3.2.5 instead of Lemma 3.2.3. The details will be omitted.

We proceed with the proof of the inequality

$$\varlimsup_{N\to\infty} \sup_{|t|\leqslant T_N} |X(t,N)|/(2\Gamma_{0N}\log\log\Gamma_{0N})^{1/2} \geqslant 1-\varepsilon \quad \text{a.s.} \tag{3.2.24}$$

for $0<\varepsilon<1/2$. Put $N_1 = 1$. Define $N_{k+1} = \min\{n : \Gamma_{0n} \geqslant (10d/\varepsilon^2)^k\}$. By Condition (ii), we have

$$(10d/\varepsilon^2)^k \leqslant \Gamma_{0N_{k+1}} < d(10d/\varepsilon^2)^k. \tag{3.2.25}$$

We observe that

$$\begin{aligned} &\varlimsup_{N\to\infty} \sup_{|t|\leqslant T_N} |X(t,N)|/(2\Gamma_{0N}\log\log\Gamma_{0N})^{1/2} \qquad (3.2.26)\\ &\geqslant \varlimsup_{k\to\infty} \sup_{|t|\leqslant T_{N_k}} |X(t,N_k)|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2} \end{aligned}$$

$$\geqslant \overline{\lim_{k\to\infty}} \sup_{|t|\leqslant T_{N_k}} |X(t,N_k)-X(t,N_{k-1})|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2}$$

$$-\overline{\lim_{k\to\infty}} \sup_{|t|\leqslant T_{N_k}} |X(t,N_{k-1})|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2}.$$

Using Lemma 3.2.5, we have

$$P\{\sup_{|t|\leqslant T_{N_k}} |X(t,N_{k-1})|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2}\geqslant \varepsilon/2\}$$

$$\leqslant C(1+T_{N_k}\Gamma_{1N_{k-1}}/\Gamma_{0N_{k-1}})\exp\{-(\varepsilon^2\Gamma_{0N_k}\log\log\Gamma_{0N_k})/5\Gamma_{0N_{k-1}}\}$$

$$\leqslant C(1+10T_{N_k}\Gamma_{1N_k}/\varepsilon^2\Gamma_{0N_k})(\log\Gamma_{0N_k})^{-2}$$

$$\leqslant ck^{-3/2} \tag{3.2.27}$$

by Condition (iii), which implies

$$\overline{\lim_{k\to\infty}} \sup_{|t|\leqslant T_{Nk}} |X(t,N_{k-1})|/(2\Gamma_{0Nk}\log\log\Gamma_{0Nk})^{1/2}\leqslant \varepsilon/2 \quad \text{a.s.} \tag{3.2.28}$$

Now we prove that

$$\overline{\lim_{k\to\infty}} \sup_{|t|\leqslant T_{Nk}} |X(t,N_k)-X(t,N_{k-1})|/(2\Gamma_{0Nk}\log\log\Gamma_{0Nk})^{1/2}\geqslant 1-\varepsilon/2 \quad \text{a.s.} \tag{3.2.29}$$

Note that $\Gamma_{0Nk}-\Gamma_{0Nk-1}\geqslant \Gamma_{0Nk}(1-\varepsilon^2/10)$. Revise the definition (3.2.13) of $\theta_N$ as follows. Define $\theta_{Nk}(\varepsilon)$ to be the solution of the equation

$$\sum_{i=Nk-1+1}^{Nk} (\gamma_i/\lambda_i)e^{-2\lambda_i\theta N_k(\varepsilon)}=\varepsilon(\Gamma_{0N_k}-\Gamma_{0N_{k-1}}).$$

Using Lemma 3.2.6 we have

$$P\{\sup_{|t|\leqslant T_{N_k}} |X(t,N_k)-X(t,N_{k-1})|/(2\Gamma_{0N_k}\log\log\Gamma_{0N_k})^{1/2}\geqslant 1-\varepsilon/2\}$$

$$\geqslant C(1+T_{N_k}/\theta_{N_k}(\frac{\varepsilon}{4}))\exp\{-(1-\frac{\varepsilon}{2})(\Gamma_{0N_k}\log\log\Gamma_{0N_k})$$

$$/(\Gamma_{0N_k}-\Gamma_{0N_{k-1}})\}$$

$$\geqslant c(\log\Gamma_{0N_k})^{-(1-\varepsilon/4)}$$

$$\geqslant ck^{-(1-\varepsilon/4)}.$$

Consequently, we obtain (3.2.29) since $\sup_{|t| \leqslant TN_k} |X(t,N_k) - X(t,N_{k-1})|, k \geqslant 1$, are independent random variables. Inserting (3.2.28) and (3.2.29) into (3.2.26) yields (3.2.24). Combining (3.2.23) and (3.2.24) implies the conclusions of Theorem 3.2.3.

## 3.3 Infinite Series

The main aim of this section is to study the path properties of the process $X(\cdot)$ defined by (3.1.3). It is obviously an a.s. finite Gaussian random variable for each fixed $t$ with mean zero and variance

$$\Gamma_0 = \sum_{i=1}^{\infty} \gamma_i / \lambda_i, \tag{3.3.1}$$

provided we assume $\Gamma_0 < \infty$. However, under the latter condition only, $X(\cdot)$ does not necessarily exist as an a.s. continuous Gaussian process in $t \in R$. We first study the existence and continuity of the process $X(\cdot)$. Then we establish the moduli of continuity and a the law logarithmic-type.

### 3.3.1 Existence and continuity

First, we have

**Theorem 3.3.1** (Csáki et al. 1990) *Assume that for some $\delta > 0$*

$$\sum_{k=1}^{\infty} \gamma_k (\log(\lambda_k \vee e))^{1+\delta} / \lambda_k < \infty . \tag{3.3.2}$$

*Then $X(t,n) \to X(t)$ uniformly in $t$ over any finite interval with probability one.*

The conclusion of Theorem 3.3.1 means that for any $\varepsilon > 0$, $T > 0$ and for almost all $\omega \in \Omega$ there exists an integer $n_0 = n_0(\varepsilon, T, \omega)$ such that

$$\sup_{|t| \leqslant T} |X(t,n,\omega) - X(t,\omega)| \leqslant \varepsilon, \tag{3.3.3}$$

whenever $n \geqslant n_0$. Therefore it follows immediately that the following conclusion is true.

**Corollary 3.3.1** *Given Condition* (3.3.2), $\{X(t), -\infty < t < \infty\}$ *is continuous with probability one.*

Proof of Theorem 3.3.1.

We prove (3.3.3). To this end, on account of the Ito-Nisio Theorem (1968), it suffices to show that

$$\sup_{|t| \leqslant T} |X(t,n) - X(t)| = \sup_{|t| \leqslant T} \Big| \sum_{k=n+1}^{\infty} X_k(t) \Big|$$

converges to zero in probability as $n \to \infty$, i.e., for any $\varepsilon > 0$

$$\lim_{n \to \infty} P\Big\{ \sup_{|t| \leqslant T} \Big| \sum_{k=n+1}^{\infty} X_k(t) \Big| > \varepsilon \Big\} = 0,$$

which is equivalent to that for any $\varepsilon > 0$ and $0 < \eta < 1$ there exists $n_0 = n_0(\varepsilon, \eta)$ such that

$$P\{ \sup_{|t| \leqslant T} |X_{mn}(t)| > \varepsilon \} \leqslant \eta \tag{3.3.4}$$

whenever $m > n \geqslant n_0$, where $X_{mn}(t) = \sum_{k=n+1}^{m} X_k(t)$. The latter process is a stationary, mean zero Gaussian process with

$$EX_{mn}^2(t) = \sum_{k=n+1}^{m} \gamma_k / \lambda_k ,$$

$$EX_{mn}(t) X_{mn}(s) = \sum_{k=n+1}^{m} (\gamma_k / \lambda_k) \exp(-\lambda_k |t-s|)$$

and

$$E(X_{mn}(t) - X_{mn}(s))^2 = 2 \sum_{k=n+1}^{m} (\gamma_k/\lambda_k)(1 - \exp(-\lambda_k |t-s|)).$$

Now we want to apply Fernique's Lemma (Lemma 1.5.1) to the process $X_{mn}(t)$ in $t$ over $[-T, T]$ with $n < m$ and $T > 0$ fixed. In order to be able to do this, we first show that under Condition (3.3.2), we have

$$\int_0^{1/e} \frac{\Big( \sum_{k=n+1}^{m} (\gamma_k/\lambda_k)(1 - \exp(-\lambda_k u)) \Big)^{1/2}}{u (\log(1/u))^{1/2}} du < \infty , \tag{3.3.5}$$

where the finiteness of the latter integral is equivalent to

$$\int_1^{\infty} \Lambda(e^{-y^2}) dy < \infty$$

in Lemma 1.5.1, where

$$\Lambda(u)=\left(2\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)(1-\exp(-\lambda_k u))\right)^{1/2}.$$

Put $K_1=\{k:\lambda_k<u^{-1/2}\}\bigcap[n+1,m]$ and $K_2=\{k:\lambda_k\geqslant u^{-1/2}\}\bigcap[n+1,m]$. Then we have

$$k\in K_1 \text{ implies } (\gamma_k/\lambda_k)(1-\exp(-\lambda_k u))\leqslant(\gamma_k/\lambda_k)u^{1/2},$$

$$k\in K_2 \text{ implies } \frac{1}{2}\log\frac{1}{u}\leqslant\log\lambda_k \text{ and so}$$

$$(\gamma_k/\lambda_k)(1-e^{-\lambda_k u})\leqslant\frac{\gamma_k}{\lambda_k}\left(\frac{2\log(\lambda_k\vee e)}{\log(1/u)}\right)^{1+\delta}$$

for any $\delta>0$. Consequently, we have

$$\begin{aligned}\Lambda^2(u)&=2\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)(1-\exp(-\lambda_k u))\\&\leqslant 2\sum_{k\in K_1}(\gamma_k/\lambda_k)u^{1/2}+2\sum_{k\in K_2}\frac{\gamma_k}{\lambda_k}\left(\frac{2\log(\lambda_k\vee e)}{\log(1/u)}\right)^{1+\delta}\\&\leqslant 2\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)(2\log(\lambda_k\vee e))^{1+\delta}(u^{1/2}+(\log(1/u))^{-(1+\delta)})\end{aligned}$$

and hence

$$\begin{aligned}&\int_0^{1/e}\frac{\Lambda(u)}{u(\log(1/u))^{1/2}}\,du\\&\leqslant\left(2\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}(2\log(\lambda_k\vee e))^{1+\delta}\right)^{1/2}\int_0^{1/e}\frac{(u^{1/2}+(\log(1/u))^{-(1+\delta)})^{1/2}}{u(\log(1/u))^{1/2}}\,du\\&=D\left(2\sum_{k=n+1}^{m}\frac{\gamma_k}{\lambda_k}(2\log(\lambda_k\vee e))^{1+\delta}\right)^{1/2},\end{aligned}\tag{3.3.6}$$

where $D$ is the finite value of the indicated integral on the right-hand side of the first inequality of (3.3.6).

Now we are ready to apply Lemma 1.5.1 to show that (3.3.4) is true.

With $a=e$, $\varepsilon=x\left(\Gamma+4\int_0^{1/e}\frac{\Lambda(u)}{u(\log(1/u))^{1/2}}\,du\right)$, where $\Gamma^2=EX^2_{nm}(t)$ $=\sum_{k=n+1}^{m}\gamma_k/\lambda_k$, by Lemma 1.5.1, (3.3.6) and (3.3.2), we obtain

$$P\{\sup_{0\leqslant t\leqslant 1} |X_{mn}(t)|>\varepsilon\}$$

$$\leqslant ce^2(1-\Phi(\varepsilon/(\Gamma+4\int_0^{1/c}\frac{\wedge(u)}{u(\log(1/u))}du))$$

$$\leqslant ce^2(1-\Phi(\varepsilon/((\sum_{k=n+1}^{m}\gamma_k/\lambda_k)^{1/2}+4D(2\sum_{k=n+1}^{m}(\gamma_k/\lambda_k)(2\log(\lambda_k\vee e))^{1+\delta})^{1/2})))$$

$$\rightarrow 0 \quad \text{as} \quad n\rightarrow\infty. \tag{3.3.7}$$

Hence, using the stationarity, we have

$$P\{\sup_{|t|\leqslant T}|X_{mn}(t)|>\varepsilon\}\leqslant 2(T+1)P\{\sup_{0\leqslant t\leqslant 1}|X_{mn}(t)|>\varepsilon\}\leqslant\eta$$

whenever $m>n\geqslant n_0$ for some large $n_0$. This also completes the proof of Theorem 3.3.1.

### 3.3.2 Moduli of continuity

We first establish some large deviation inequalities on which continuity moduli results are based.

**Lemma 3.3.1** *Assume that Condition* (3.3.2) *is satisfied and that* $\sigma(\cdot)$ *defined by*

$$\sigma^2(s)=E(X(t+s)-X(t))^2, \tag{3.3.8}$$

*is a regular varying function at zero with a positive exponent, namely*

$$\sigma(s)=s^{\alpha}L(s), \qquad \alpha>0, \tag{3.3.9}$$

*where* $L(\cdot)$ *is a slowly varying at zero, i.e. it is measurable, positive and*

$$\lim_{s\rightarrow 0}L(\lambda s)/L(s)=1 \qquad \text{for all } \lambda>0.$$

*Then for any* $\varepsilon>0$ *there exist constants* $C=C(\varepsilon)>0$ *and* $0<h(\varepsilon)<1$ *such that*

$$P\{\sup_{0\leqslant t\leqslant 1-h}\sup_{0\leqslant s\leqslant h}|X(t+s)-X(t)|\geqslant v\sigma(h)\}\leqslant\frac{C}{h}\exp\left(-\frac{v^2}{2+\varepsilon}\right) \tag{3.3.10}$$

*for every* $v>0$ *and* $0<h<h(\varepsilon)$.

*Proof* For any positive real numbers $t$ and $r$ we let $t_r = [2^r t]/2^r$ and write also $R = 2^r$. Clearly, using the continuity of $X(\cdot)$, we have

$$|X(t+s) - X(t)| \leqslant |X((t+s)_r) - X(t_r)| + \sum_{j=0}^{\infty} |X((t+s)_{r+j+1}) - X((t+s)_{r+j})|$$

$$+ \sum_{j=0}^{\infty} |X(t_{r+j+1}) - X(t_{r+j})|. \tag{3.3.11}$$

we have for $0 < h < 1$ and $u > 0$

$$P\{\sup_{0\leqslant t\leqslant 1-h}\ \sup_{0\leqslant s\leqslant h} |X((t+s)_r) - X(t_r)| \geqslant u\sigma(h+R^{-1})\} \leqslant 2R(Rh+1)e^{-u^2/2}, \tag{3.3.12}$$

$$P\{\sup_{0\leqslant t\leqslant 1-h}\ \sup_{0\leqslant s\leqslant h} |X((t+s)_{r+j+1}) - X((t+s)_{r+j})| \geqslant (u^2+2j)^{1/2}\sigma(2^{-(r+j+1)})\}$$

$$\leqslant 2\exp\left(-\frac{u^2+2j}{2}\right)\cdot 2^{r+j+1} \leqslant 4Re^{-u^2/2}(2/e)^j, \tag{3.3.13}$$

as well as

$$P\{\sup_{0\leqslant t\leqslant 1-h}\ \sup_{0\leqslant s\leqslant h} |X(t_{r+j+1}) - X(t_{r+j})| \geqslant (u^2+2j)^{1/2}\sigma(2^{-(r+j+1)})\}$$

$$\leqslant 4Re^{-u^2/2}(2/e)^j. \tag{3.3.14}$$

Thus

$$P\{\sup_{0\leqslant t\leqslant 1-h}\ \sup_{0\leqslant s\leqslant h} |X(t+s) - X(t)|$$

$$\geqslant u\sigma(h+R^{-1}) + 2\sum_{j=0}^{\infty}(u^2+2j)^{1/2}\sigma(2^{-(r+j+1)})\}$$

$$\leqslant (2R(Rh+1) + 8RD)e^{-u^2/2}, \tag{3.3.15}$$

where $D = \sum_{j=0}^{\infty}(2/e)^j$. Let $R$ be such that $2R > A/h \geqslant R$, where $A$ is a positive constant to be specified later on. Since $L(\cdot)$ is a slowly varying function, for any given $0 < \varepsilon < 1$ taking $r$ large enough (equivalently $h$ small enough), we have

$$\sigma(2^{-(r+j+1)}) = 2^{-(r+j+1)\alpha}L(2^{-(r+j+1)}) \leqslant (1+\varepsilon)2^{-(r+j+1)\alpha}L(2^{-(r+j)})$$

$$\leqslant (1+\varepsilon)^{j+1}2^{-(r+j+1)\alpha}L(2^{-r}) \leqslant 2^{-(j+1)\alpha/2}(h\frac{1}{hR})^{\alpha}L(h\frac{1}{hR})$$

$$\leqslant 2^{-(j+1)\alpha/2}(\frac{2}{A})^{\alpha}(1+\frac{\varepsilon}{9})h^{\alpha}L(h) = (1+\frac{\varepsilon}{9})(\frac{2}{A})^{\alpha}2^{-(j+1)\alpha/2}\sigma(h).$$

Then we have

$$u\sigma(h+R^{-1}) + 2\sum_{j=0}^{\infty}(u^2+2j)^{1/2}\sigma(2^{-(r+j+1)}) \tag{3.3.16}$$

$$\leqslant u\sigma(h)(1+\frac{\varepsilon}{9})(1+\frac{2}{A})^{\alpha} + 2(\frac{2}{A})^{\alpha}(1+\frac{\varepsilon}{9})\sum_{j=0}^{\infty}(u^2+2j)^{1/2}2^{-(j+1)\alpha/2}\sigma(h)$$

$$\leqslant u\sigma(h)(1+\frac{\varepsilon}{9})[(1+\frac{2}{A})^{\alpha} + 2(\frac{2}{A})^{\alpha}\sum_{j=0}^{\infty}2^{-(j+1)\alpha/2}]$$

$$+2(\frac{2}{A})^{\alpha}(1+\frac{\varepsilon}{9})\sigma(h)\sum_{j=0}^{\infty}(2j)^{1/2}2^{-(j+1)\alpha/2}.$$

Let $u = v/(1+\varepsilon/8)$. Since, without loss of generality, we can assume $v \geqslant 1$, the right-hand side of (3.3.16) does not exceed

$$u\,\sigma(h)(1+\varepsilon/8) = v\sigma(h)$$

provided that $A$ is large enough. Moreover

$$(2R(Rh+1)+8RD)e^{-u^2/2} \leqslant (D'A/h)e^{-v^2/(2+\varepsilon)} \tag{3.3.17}$$

with $D' = 2(A+1)+8D$. Consequently (3.3.10) follows from (3.3.15).

*Remark* 3.3.1 The proof of the lemma does not use the concrete distribution of the process. Hence, in fact it is true for any a.s. continuous Gaussian process with means zero and $\sigma^2(\cdot)$ of (3.3.8) satisfying (3.3.9) (cf. Csáki, Csörgő, Lin and Révész 1990).

*Remark* 3.3.2 We have a version of (3.3.10) as follows:

$$P\{\sup_{|t|\leqslant T}\sup_{0\leqslant s\leqslant h}|X(t+s)-X(t)| \geqslant v\sigma(h)\} \leqslant \frac{CT}{h}\exp(-\frac{v^2}{2+\varepsilon})$$

**Theorem 3.3.2** (Csáki et al. 1990) *Assume that* $\{X(t)\}$ *satisfies the conditions in Lemma* 3.3.1. *Then we have*

$$\lim_{h \downarrow 0} \sup_{0 \leqslant t \leqslant 1-h} \sup_{0 \leqslant s \leqslant h} \frac{|X(t+s)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}} = 1 \quad \text{a.s.} \qquad (3.3.18)$$

$$\lim_{h \downarrow 0} \lim_{0 \leqslant t \leqslant 1-h} \frac{|X(t+h)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}} = 1 \quad \text{a.s.} \qquad (3.3.19)$$

*Proof* Using Lemma 3.3.1, in the same way as the proof of the first part of the Lévy modulus of continuity in Csörgő and Révész (1981) we can prove

$$\overline{\lim_{h \downarrow 0}} \sup_{0 \leqslant t \leqslant 1-h} \sup_{0 \leqslant s \leqslant h} \frac{|X(t+s)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}} \leqslant 1 \quad \text{a.s.} \qquad (3.3.20)$$

Consequently, in order to prove this theorem, it sufficies to show that

$$\underline{\lim_{h \downarrow 0}} \sup_{0 \leqslant t \leqslant 1-h} \frac{|X(t+h)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}} \geqslant 1 \quad \text{a.s.} \qquad (3.3.21)$$

For $0 \leqslant i < j$,

$$E(X((i+1)h)-X(ih))(X((j+1)h)-X(jh)) \qquad (3.3.22)$$
$$= \sum_{k=0}^{\infty} (\gamma_k/\lambda_k) e^{-\lambda_k (j-i)}(2-e-e^{-1}) < 0.$$

Hence, by Lemma 1.1.1, we have

$$P\{ \sup_{0 \leqslant t \leqslant 1-h} \frac{|X(t+h)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}} \leqslant (1-\varepsilon)^{1/2} \} \qquad (3.3.23)$$

$$\leqslant P\{ \max_{0 \leqslant i \leqslant 1/h-1} \frac{X((i+1)h)-X(ih)}{\sigma(h)} \leqslant (2(1-\varepsilon)\log(1/h))^{1/2} \}$$

$$\leqslant (1 - \frac{h^{1-\varepsilon}}{(16\pi\log(1/h))^{1/2}})^{[1/h]}$$

$$\leqslant \exp\{ - \frac{h^{-\varepsilon}}{(16\pi\log(1/h))^{1/2}} \}.$$

Let now $h = h_n = 1/n$. Then the last inequality implies

$$\sum_{n=1}^{\infty} P\{ \sup_{0 \leqslant t \leqslant 1-1/n} \frac{|X(t+1/n)-X(t)|}{\sigma(1/n)(2\log n)^{1/2}} \leqslant (1-\varepsilon)^{1/2} \} < \infty ,$$

and by the Borel-Cantelli Lemma, we conclude

$$\varlimsup_{n \to \infty} \sup_{0 \leqslant t \leqslant 1-1/n} \frac{|X(t+1/n)-X(t)|}{\sigma(1/n)(2\log n)^{1/2}} \geqslant (1-\varepsilon)^{1/2} \qquad \text{a.s.} \qquad (3.3.24)$$

Considering $1/(n+1) < h \leqslant 1/n$, we have

$$\sup_{0 \leqslant t \leqslant 1-h} \frac{|X(t+h)-X(t)|}{\sigma(h)(2\log(1/h))^{1/2}}$$

$$\geqslant \sup_{0 \leqslant t \leqslant 1-1/n} \frac{|X(t+1/n)-X(t)|}{\sigma(1/n)(2\log n)^{1/2}} \cdot \frac{\sigma(1/n)(2\log n)^{1/2}}{\sigma(h)(2\log(1/h))^{1/2}}$$

$$-2 \sup_{0 \leqslant t \leqslant 1-\frac{1}{n(n+1)}} \sup_{0 \leqslant s \leqslant \frac{1}{n(n+1)}} \frac{|X(t+s)-X(t)|}{\sigma(1/n(n+1))(2\log n(n+1))^{1/2}}$$

$$\cdot \frac{\sigma(1/n(n+1))(2\log n(n+1))^{1/2}}{\sigma(h)(2\log(1/h))^{1/2}} .$$

By regular variation of $\sigma(\cdot)$ at zero, with $1/(n+1) < h \leqslant 1/n$ we get

$$\lim_{n \to \infty} \sigma(1/n)(2\log n)^{1/2} / (\sigma(h)(2\log(1/h))^{1/2}) = 1 \qquad (3.3.25)$$

and

$$\lim_{n \to \infty} \sigma\left(\frac{1}{n(n+1)}\right)(2\log n(n+1))^{1/2} / \sigma(h)(2\log(1/h))^{1/2} = 0 \qquad (3.3.26)$$

Consequently, the above inequality implies that we have (3.3.21) by (3.3.24) and (3.3.20) combined with (3.3.25) and (3.3.26) respectively. This also concludes the proof of Theorem 3.3.2.

*Remark* 3.3.3 Recalling Remark 3.3.2, we can rewrite (3.3.18) and (3.3.19) as follows: For $T_h \uparrow \infty$ continuously as $h \to 0$,

$$\lim_{h \downarrow 0} \sup_{|t| \leqslant T_h} \sup_{0 \leqslant s \leqslant h} \frac{|X(t+s)-X(t)|}{\sigma(h)(2\log(T_k/h))^{1/2}} = 1 \qquad \text{a.s.} \qquad (3.3.27)$$

$$\lim_{h \downarrow 0} \sup_{|t| \leqslant T_h} \frac{|X(t+h)-X(t)|}{\sigma(h)(2\log(T_h/h))^{1/2}} = 1 \qquad \text{a.s.} \qquad (3.3.28)$$

(Csörgő and Lin 1990b).

### 3.3.3 Another version of continuity modulus

Csörgő and Révész (1981 Theorem 1.3.3) showed that another

continuity modulus theorem for a Wiener process, i.e.

$$\overline{\lim_{t\to 0}} \frac{|W(t)|}{(2t\log\log(1/t))^{1/2}} = 1 \quad \text{a.s.}$$

$$\lim_{t\to 0} \sup_{0\leqslant s\leqslant t} \frac{|W(s)|}{(2s\log\log(1/s))^{1/2}} = 1 \quad \text{a.s.}$$

They pointed out that it can be proved by the substitution

$$\widetilde{W}(t) = \begin{cases} tW(1/t) & \text{if} \quad t>0, \\ 0 & \text{if} \quad t=0. \end{cases}$$

But it is only true for the first equality, not true for the latter. The latter equality can be proved directly. A similar continuity mudulus theorem holds true for the process $\{X(t)\}$.

**Theorem 3.3.3** (Lu et al. 1991) *Suppose that the process* $\{X(t)\}$ *satisfies Conditions* (3.3.1), (3.3.2) *and*

$$\Gamma_1 = \sum_{i=1}^{\infty} \gamma_i < \infty. \tag{3.3.29}$$

*Then we have*

$$\overline{\lim_{h\to 0}} \sup_{0\leqslant s\leqslant h} \frac{|X(s)-X(0)|}{\sigma(h)(2\log\log(1/h))^{1/2}} = 1 \quad \text{a.s.} \tag{3.3.30}$$

$$\overline{\lim_{h\to 0}} \frac{|X(h)-X(0)|}{\sigma(h)(2\log\log(1/h))^{1/2}} = 1 \quad \text{a.s.} \tag{3.3.31}$$

*Proof* First, we prove

$$\overline{\lim_{h\to 0}} \sup_{0\leqslant s\leqslant h} \frac{|X(s)-X(0)|}{\sigma(h)(2\log\log(1/h))^{1/2}} \leqslant 1 \quad \text{a.s.} \tag{3.3.32}$$

Let $Y(s)=X(sh)-X(0), 0\leqslant s\leqslant 1$. Then $\{Y(s); 0\leqslant s\leqslant 1\}$ is a Gaussian process with $EY(s)=0$ and

$$\begin{aligned} E|Y(t)-Y(s)|^2 &= 2\sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i}(1-\exp(-h|t-s|\lambda_i)) \qquad (3.3.33) \\ &\leqslant 2\Gamma_1 h|t-s| =: \Lambda^2(|t-s|), \end{aligned}$$

where $\Lambda(x)=(2x\Gamma_1 h)^{1/2}$ satisfying the condition of Lemma 1.5.1. And for $0\leqslant t\leqslant 1$

$$E\,|\,Y^2(t)|=2\sum_{i=1}^{\infty}\frac{\gamma_i}{\lambda_i}\,(1-\exp(-th\lambda_i))\leqslant 2\Gamma_1 h=:\Gamma^2\ (\Gamma>0). \qquad (3.3.34)$$

By (3.3.29), we have

$$\lim_{h\to 0}\sigma^2(h)/2\Gamma_1 h=1.$$

Hence, from the monotonicity of function $\sigma(h)$, for any $\varepsilon>0$, there exists an $h_0>0$ such that for any $0\leqslant h\leqslant h_0$

$$(1-\varepsilon/2)2\Gamma_1 h\leqslant\sigma^2(h)\leqslant 2\Gamma_1 h. \qquad (3.3.35)$$

It is clear that for any $\varepsilon>0$

$$4\int_1^{\infty}\Lambda(a^{-u^2})du=4\sqrt{2\Gamma_1 h}\int_1^{\infty}a^{-u^2/2}du\leqslant\varepsilon\sqrt{2\Gamma_1 h} \qquad (3.3.36)$$

for a large enough . By (3.3.33)-(3.3.36) and using Lemma 1.5.1, we have

$$P\{\sup_{0\leqslant s\leqslant h}|X(s)-X(0)|/(\sigma(h)(2\log\log(1/h))^{1/2})\geqslant(1+\varepsilon)^3\}$$

$$\leqslant P\{\sup_{0\leqslant t\leqslant 1}|Y(t)|\geqslant(1+\varepsilon)(\sqrt{2\Gamma_1 h}$$

$$+4\int_1^{\infty}\Lambda(a^{-u^2})\int_1^{\infty}\Lambda(a^{-u^2}))du\sqrt{2\log\log(1/h)}\}$$

$$\leqslant ca^2\int_{(1+\varepsilon)\sqrt{2\log\log(1/h)}}^{\infty}\exp\left(-\frac{u^2}{2}\right)du$$

$$\leqslant ca^2(\log(1/h))^{1+\varepsilon}.$$

Put $h_k=\theta^{-k}$, $\theta>1$. It follows from the Borel-Cantelli lemma that

$$\overline{\lim_{k\to\infty}}\sup_{0\leqslant s\leqslant h_k}\frac{|X(s)-X(0)|}{\sigma(h_k)(2\log\log(1/h_k))^{1/2}}\leqslant(1+\varepsilon)^3 \quad \text{a.s.}$$

For any given $h>0$, there exists a $k>0$ such that $h_{k+1}\leqslant h\leqslant h_k$. Note that

$$\overline{\lim_{h\to 0}}\sup_{0\leqslant s\leqslant h}\frac{|X(s)-X(0)|}{\sigma(h)(2\log\log(1/h))^{1/2}}$$

$$\leqslant \varlimsup_{k\to\infty} \sup_{0\leqslant s\leqslant h_k} \frac{|X(s)-X(0)|}{\sigma(h_k)(2\log\log(1/h_k))^{1/2}} \frac{\sigma(h_k)}{\sigma(h_{k+1})}$$

$$\leqslant (1+\varepsilon)^3\theta^{1/2}.$$

which proves (3.3.32) by letting $\varepsilon\to 0$ and $\theta\to 1$.

Next, we prove

$$\varlimsup_{h\to 0} |X(h)-X(0)|/\sigma(h)(2\log\log(1/h))^{1/2})\geqslant 1 \qquad \text{a.s.} \tag{3.3.37}$$

Put $h_n = e^{-n}$,

$$Y_n = (X(h_n)-X(0))/(\sigma(h_n), \quad A_n = \{\, Y_n \geqslant (1-\varepsilon)(2\log\log(1/h_n))^{1/2}\,\}$$

By elementary calculation, it is easy to see that

$$\sum_{n=1}^{\infty} P(A_n) = \infty\ . \tag{3.3.38}$$

In order to prove (3.3.37), we need only to prove $P\{\,A_n\ i.\ o.\,\}=1$ ,and it suffices by (3. 338) to check the Condition (ii) of Lemma 1. 5. 4. From (3.3.35), there exists an $N_0$ such that

$$(1-\varepsilon)2\Gamma_1 h_n \leqslant \sigma^2(h_n) \leqslant 2\Gamma_1 h_n \qquad \text{for } n\geqslant N_0\,. \tag{3.3.39}$$

Write

$$\begin{aligned} I_n &= \sum_{1\leqslant j<k\leqslant n} \{\, P(A_j\ A_k) - P(A_j)P(A_k)\,\} \\ &= \sum_{j=1}^{N_1-1} \sum_{k=j+1}^{n} \{\, P(A_j A_k) - P(A_j)P(A_k)\} \\ &\quad + \sum_{k=N_1+1}^{n} \sum_{j=N_1}^{k-1} \{\, P(A_j A_k) - P(A_j)P(A_k)\} \\ &=: I_{1n}(N_1) + I_{2n}(N_1), \end{aligned}$$

where $N_1$ will be defined later on.

Let $N_2\geqslant N_0$ , denote $u_k=\log k$. By Lemma 1.5.3.

$$|I_{2n}(N_2)| \leqslant \sum_{k=N_2+1}^{n} \sum_{j=N_2}^{k-1} |r_{jk}|\,\varphi(\lambda_j, \lambda_k; r_{jk}^*) \tag{3.3.40}$$

$$= \Big( \sum_{k=N_2+1}^{n} \sum_{j=N_2}^{k-u_k} + \sum_{k=N_2+1}^{n} \sum_{j=k-u_k+1}^{k-1} \Big) \frac{|r_{jk}|}{2\pi(1-r_{jk}^{*2})^{1/2}} \exp\Big(-\frac{\lambda_j^2+\lambda_k^2-2\lambda_j\lambda_k r_{jk}^*}{2(1-r_{jk}^{*2})}\Big)$$

$$=: J_{1n}(N_2) + J_{2n}(N_2)$$

where $r_{jk} = EY_j Y_k$, $\lambda_j = (1-\varepsilon)(2\log\log(1/h_j))^{1/2} = (1-\varepsilon)(2\log j)^{1/2}$, $0 \leqslant r^*_{jk} \leqslant r_{jk}$. From (3.3.39), we have

$$0 < r_{jk} = E(X(h_k) - X(0))(X(h_j) - X(0))/\sigma(h_k)\sigma(h_j) \tag{3.3.41}$$

$$\leqslant \sigma(h_k)/\sigma(h_j) \leqslant ((1-\varepsilon)e)^{-1} =: r,$$

therefore

$$J_{1n}(N_2) \leqslant \sum_{k=N_2+1}^{n} \sum_{j=N_2}^{k-u_k} \frac{r_{jk}\lambda_j\lambda_k}{(1-r^2)^{1/2}} \psi(\lambda_j)\psi(\lambda_k) \tag{3.3.42}$$

$$\exp\left\{ -\frac{1}{2(1-r^{*2}_{jk})} (r^*_{jk}(\lambda_j^2 + \lambda_k^2) - 2\lambda_k\lambda_j r^*_{jk})\right\}$$

$$\leqslant \sum_{k=N_2+1}^{n} \sum_{j=N_2}^{k-u_k} \frac{r_{jk}\lambda_k^2}{(1-r^2)^{1/2}} \exp(r_{jk}\lambda_k^2)\psi(\lambda_j)\psi(\lambda_k)$$

where $\psi(x) = \dfrac{1}{x\sqrt{2\pi}} \exp(-x^2/2)$. By (3.3.41)

$$r_{jk} \leqslant (1/(1-\varepsilon)e^{(k-j)})^{1/2} \leqslant (1/k(1-\varepsilon))^{1/2},$$

$$r_{jk}\lambda_k^2 \leqslant (c\log k)/k^{1/2}, \qquad \text{for } k-j \geqslant u_k = \log k.$$

Hence, for any given $\delta > 0$ there exists an $N_3$ such that

$$r_{jk}\lambda_k^2 \exp(r_{jk}\lambda_k^2) < \delta(1-r^2)^{1/2} \quad \text{for} \quad k > N_3. \tag{3.3.43}$$

Combining it with (3.3.42) and letting $N_1 = N_2 \vee N_3$, we get

$$J_{1n}(N_1) \leqslant \delta\left(\sum_{i=1}^{n} P(A_i)\right)^2. \tag{3.3.44}$$

For $k - u_k + 1 \leqslant j \leqslant k-1$ and $j$ large enough we have

$$j+1 \leqslant k \leqslant j + 2\log j, \tag{3.3.45}$$

therefore we obtain

$$J_{2n}(N_2) \leqslant \sum_{j=N_1-u_{N_1}+1}^{n-1} \sum_{k=j+1}^{j+2\log j} \frac{r_{jk}\lambda_j\psi(\lambda_j)}{(2\pi(1-r^2))^{1/2}} \exp\left\{-\frac{(\lambda_k - r^*_{jk}\lambda_j)^2}{2(1-r^{*2}_{jk})}\right\}$$

$$\leqslant \sum_{j=N_1-u_{N_1}+1}^{n-1} (2\log j)c(2\log j)^{1/2}\psi(\lambda_j) j^{-\frac{1-r}{1+r}(1-\varepsilon)^2} \tag{3.3.46}$$

$$\leqslant c \sum_{j=1}^{n} P(A_j).$$

Combining (3.3.40) with (3.3.44) and (3.3.46), we obtain

$$|I_{2n}(N_1)| \leqslant \delta \left(\sum_{i=1}^{n} P(A_i)\right)^2 + c \sum_{i=1}^{n} P(A_i).$$

On the other hand

$$I_{1n}(N_1) \leqslant 2N_1 \sum_{j=1}^{n} P(A_j).$$

Thus condition (ii) of Lemma 1.5.4 is satisfied. The proof of Theorem 3.3.3 is completed.

## 3.4 $l^2$-Norm Squared Process

A $l^2$-norm squared process $\chi^2(\cdot)$ defined by

$$\chi^2(t) = \| Y(t) \|^2 = \sum_{k=1}^{\infty} X_k^2(t), \qquad -\infty < t < \infty$$

is closely related to infinite-dimensional OU process $\{ Y(t), -\infty < t < \infty \} = \{ X_k(t), -\infty < t < \infty \}_{k=1}^{\infty}$. Some papers are devoted to study its path properties. Since $\chi^2(\cdot)$ is the non-Gaussian process, we cannot employ some effective propositions (e.g. Fernique's Lemma and Slepian's Lemma), which are frequently used for a Gaussian process, to prove this kind of properties.

In addition to $\Gamma_0 < \infty$, we need also the condition $\Gamma_2 = \sum_{k=1}^{\infty} \gamma_k^2 / \lambda_k < \infty$.

### 3.4.1 Moduli of continuity

In order to establish the moduli of continuity we need some large deviation results which may be of interest in their own. Put

$$M = \max_{j \geqslant 1} \gamma_j^2 / \lambda_j .$$

**Lemma 3.4.1** *Assume* $\Gamma_0 < \infty$ *and* $\Gamma_2 < \infty$. *Then for any* $\varepsilon > 0$ *there exist* $h(\varepsilon) > 0$ *and* $C = C(\varepsilon) > 0$ *such that for any* $T > h(\varepsilon), h < h(\varepsilon)$, $t \geqslant 0$ *we have*

$$P\{ |\chi^2(t+h) - \chi^2(t)| \geqslant v(8hM)^{1/2} \} \geqslant \frac{1}{7v} \exp\left(-\frac{v}{1-\varepsilon}\right) \tag{3.4.1}$$

$$P\{ \sup_{|t| \leqslant T} \sup_{0 \leqslant s \leqslant h} |\chi^2(t+s) - \chi^2(t)| \geqslant v(8hM)^{1/2} \} \leqslant \frac{CT}{h} \exp\left(-\frac{v}{1-\varepsilon}\right) \tag{3.4.2}$$

*for any* $v \geqslant (8/\varepsilon^2)(\Gamma_2/M)^{1/2}$.

*Proof* Put $M_n = \max_{1 \leqslant j \leqslant n} \gamma_j^2/\lambda_j$, $\sigma_k^2 = E(X_k(t+h) + X_k(t))^2$ and

$\sigma_k'^2 = E(X_k(t+h) - X_k(t))^2$. Then

$$E(X_k^2(t+h) - X_k^2(t))^2 = \sigma_k^2 \sigma_k'^2 = 4(\gamma_k/\lambda_k)^2(1 - \exp(-2\lambda_k h)). \tag{3.4.3}$$

At first, we let

$$p_n(v) = P\{ | \sum_{j=1}^{n} (X_j^2(t+h) - X_j^2(t))| \geqslant v(8hM_n)^{1/2} \}$$

and prove that for large $n$

$$\frac{1}{7v} \exp\left(-\frac{v}{1-\varepsilon}\right) \leqslant p_n(v) \leqslant 2\exp\left(-\frac{v}{1+\varepsilon}\right), \tag{3.4.4}$$

provided $v \geqslant (8/\varepsilon^2)(\Gamma_2/M_n)^{1/2}$.

Let $k_0$ be an integer such that $\gamma_{k_0}^2/\lambda_{k_0} = M_n$. We put

$$Y = \sum_{j=1}^{n} (X_j^2(t+h) - X_j^2(t)) - (X_{k_0}^2(t+h) - X_{k_0}^2(t))$$

and note that $Y$ is independent of $X_{k_0}^2(t+h) - X_{k_0}^2(t)$. Since

$$\sum_{j=1}^{n} (X_j^2(t+h) - X_j^2(t)) = \sum_{j=1}^{n} (X_j(t+h) + X_j(t))(X_j(t+h) - X_j(t))$$

is symmetric, we have

$$p_n(v) = 2P\{ \sum_{j=1}^{n} (X_j^2(t+h) - X_j^2(t)) \geqslant v(8hM_n)^{1/2} \} \tag{3.4.5}$$

$$\geqslant 2P\{ X^2_{k_0}(t+h) - X^2_{k_0}(t) \geqslant v(8hM_n)^{1/2},\ Y \geqslant 0 \}$$

$$= 2P\{ X^2_{k_0}(t+h) - X^2_{k_0}(t) \geqslant v(8hM_n)^{1/2} \} P(Y \geqslant 0)$$

$$\geqslant P\{ X^2_{k_0}(t+h) - X^2_{k_0}(t) \geqslant v(8hM_n)^{1/2} \}.$$

Now we estimate $P\{ X^2_k(t+h) - X^2_k(t) \geqslant v\sigma_k \sigma_k' \}$. Let $f_k$ denote the density function of $X^2_k(t+h) - X^2_k(t)$. By the independence of $X_k(t+h) + X_k(t)$ and $X_k(t+h) - X_k(t)$ we have

$$f_k(x) = \frac{1}{\pi\sigma_k \sigma_k'} \int_0^\infty \frac{1}{y} \exp\left\{ -\frac{x^2}{2\sigma_k^2 y^2} - \frac{y^2}{2\sigma_k'^2} \right\} dy,$$

and using the tail probability estimates of the normal distribution we obtain

$$P\{ X^2_k(t+h) - X^2_k(t) \geqslant v\sigma_k\sigma_k' \} \tag{3.4.6}$$

$$= \frac{1}{\pi\sigma_k\sigma_k'} \int_0^\infty \frac{1}{y} \left( \int_{v\sigma_k\sigma_k'}^\infty \exp\left\{ -\frac{x^2}{2\sigma_k^2 y^2} \right\} dx \right) \exp\left\{ -\frac{y^2}{2\sigma_k'^2} \right\} dy$$

$$\geqslant \frac{1}{\pi\sigma_k'^2 v} \int_0^\infty y\left(1 - \frac{y^2}{v^2\sigma_k'^2}\right) \exp\left\{ -\frac{v^2\sigma_k'^2}{2y^2} - \frac{y^2}{2\sigma_k'^2} \right\} dy$$

$$\geqslant \frac{1}{\pi v} \int_{v^{1/2}}^{v^{3/4}} y\left(1 - \frac{y^2}{v^2}\right) \exp\left\{ -\frac{v^2}{2y^2} - \frac{y^2}{2} \right\} dy$$

$$\geqslant \frac{e^v}{\pi v^{1/2}} \int_{2v^{1/2}}^{v^{3/4}+v^{1/4}} \exp(-x^2/2) dx \qquad (x = y + v/y)$$

$$\geqslant \frac{e^v}{\pi v^{1/2}} \left\{ \left( \frac{1}{2v^{1/2}} - \frac{1}{8v^{3/2}} \right) \exp(-2v) \right.$$

$$\left. - \frac{1}{v^{3/4} + v^{1/4}} \exp\{ -(v^{3/4} + v^{1/4})^2/2 \} \right\}$$

$$\geqslant \frac{1}{7v} \exp(-v),$$

provided that $v$ is large enough. Since

$$\sigma_{k_0}^2 \sigma_{k_0}'^2 / (8hM_n) \to 1 \qquad \text{as} \quad h \to 0,$$

by (3.4.5) and (3.4.6) we get the left-hand side inequality of (3.4.4).

Now we prove the right-hand side inequality of (3.4.4). For $0 \leqslant x \leqslant 1/(\sigma_j \sigma_j')$ we have

$$E \exp \{ x (X_j^2(t+h) - X_j^2(t)) \}$$
$$= E \{ E [\exp (x (X_j(t+h) + X_j(t))(X_j(t+h)$$
$$- X_j(t))) \mid X_j(t+h) + X_j(t)] \}$$
$$= E \exp \{ \frac{1}{2} x^2 (X_j(t+h) + X_j(t))^2 \sigma_j'^2 \}$$
$$= (1 - x^2 \sigma_j'^2 \sigma_j^2)^{-1/2}.$$

Consequently, for $0 \leqslant x \leqslant 1/(\sigma_{k_0} \sigma_{k0}')$, we have

$$p_n(v) \leqslant 2 \exp \{ -xv (8hM_n)^{1/2} \} \prod_{j=1}^{n} (1 - x^2 \sigma_j^2 \sigma_j'^2)^{-1/2}. \quad (3.4.7)$$

Let $x = (1 - \varepsilon/2)/(\sigma_{k_0} \sigma_{k0}')$. Then

$$x^2 \sigma_{k0}^2 \sigma_{k0}'^2 \leqslant (1 - \varepsilon/2)^2 \leqslant 1 - 3\varepsilon/4,$$

and by using the inequality

$$1 - y \geqslant e^{-y/\varepsilon} \quad (3.4.8)$$

for $0 \leqslant y \leqslant 1 - \varepsilon$ and $0 < \varepsilon < 1$, we get

$$\prod_{j=1}^{n} (1 - x^2 \sigma_j^2 \sigma_j'^2)^{-1/2} \leqslant \exp \{ \frac{1}{2} \cdot \frac{4}{3\varepsilon} x^2 \sum_{j=1}^{n} \sigma_j^2 \sigma_j'^2 \}$$
$$\leqslant \exp \{ \frac{1}{\varepsilon} \sum_{j=1}^{n} \sigma_j^2 \sigma_j'^2 / (\sigma_{k_0}^2 \sigma_{k_0}'^2) \}.$$

Hence, on assuming that $v \geqslant (8/\varepsilon^2)(\Gamma_2/M_n)^{1/2}$, by (3.4.7) we obtain

$$p_n(v) \leqslant 2\exp \{ -(1 - \varepsilon/2) v (8hM_n)^{1/2} / (\sigma_{k0} \sigma_{k_0}') + \frac{1}{\varepsilon} \sum_{j=1}^{n} \sigma_j^2 \sigma_j'^2 / (\sigma_{k_0}^2 \sigma_{k_0}'^2) \}$$
$$\leqslant 2 \exp \{ -(1 - 2\varepsilon/3) v \} \leqslant 2\exp ( - \frac{v}{1+\varepsilon} ),$$

provided that $h$ is small enough. This also completes the proof of (3.4.4).

By assuming $\Gamma_2 < \infty$, $M_n = M$ for all large $n$. For all such $n$ (3.4.4) remains true when $M_n$ is replaced by $M$ in the definition of $p_n(v)$.

Consequently, (3.4.4) yields

$$\frac{1}{7v}\exp\left(-\frac{v}{1-\varepsilon}\right)\leqslant P\{\,|\chi^2(t+h)-\chi^2(t)|\geqslant v(8hM)^{1/2}\}\leqslant 2\exp\left(-\frac{v}{1+\varepsilon}\right) \tag{3.4.9}$$

if $v\geqslant(8/\varepsilon^2)(\Gamma_2/M)^{1/2}$. The left-hand side inequality of (3.4.9) is (3.4.1), while (3.4.2) can be proved along the lines of the proof of Lemma 3.3.2 with the help of the right-hand side inequality of (3.4.9). Hence we omit these details.

**Theorem 3.4.1** (Csörgő and Lin 1990b) *Assume* $\Gamma_0<\infty$ *and* $\Gamma_2<\infty$. *Then for any* $T_h\uparrow\infty$ *continuously as* $h\to 0$,

$$\lim_{h\downarrow 0}\ \sup_{|t|\leqslant Th}\ \sup_{0\leqslant s\leqslant h}\frac{|\chi^2(t+s)-\chi^2(t)|}{(8hM)^{1/2}\log(T_h/h)}\leqslant 1 \qquad \text{a.s.}$$

*If, in addition, the continuous, non-decreasing function* $T_h$ *satisfies also*

$$\log T_h/\log(1/h)\to\infty \quad \text{as} \quad h\to 0, \tag{3.4.10}$$

*then we have as well*

$$\lim_{h\downarrow 0}\ \sup_{|t|\leqslant Th}\ \sup_{0\leqslant s\leqslant h}\frac{|\chi^2(t+s)-\chi^2(t)|}{(8hM)^{1/2}\log(T_h/h)}=1 \qquad \text{a.s.}$$

$$\lim_{h\downarrow 0}\ \sup_{|t|\leqslant Th}\frac{|\chi^2(t+h)-\chi^2(t)|}{(8hM)^{1/2}\log(T_h/h)}=1 \qquad \text{a.s.}$$

*Proof* For given $0<\varepsilon<1$, let $h_n$ be such that

$$\sum_{n=1}^{\infty}(h_{n-1}/T_{hn})^{\varepsilon/2}<\infty \tag{3.4.11}$$

and as $n\to\infty$,

$$(h_{n-1}/T_{hn})/(h_n/T_{hn+1})\to 1. \tag{3.4.12}$$

By Lemma 3.4.1 we have

$$P\{\sup_{|t|\leqslant Thn}\ \sup_{0\leqslant s\leqslant hn-1}|\chi^2(t+s)-\chi^2(t)|\geqslant(1+2\varepsilon)(8h_{n-1}M)^{1/2}\log(T_{hn}/h_{n-1})\}$$

$$\leqslant(CT_{hn}/h_{n-1})\exp\left\{-\frac{1+2\varepsilon}{1+\varepsilon}\log(T_{hn}/h_{n-1})\right\}\leqslant C(h_{n-1}/T_{hn})^{\varepsilon/2}.$$

The latter combined with (3.4.11) yields

$$\overline{\lim_{n\to\infty}} \sup_{|t|\leqslant T_{h_n}} \sup_{0\leqslant s\leqslant h_{n-1}} \frac{|\chi^2(t+s)-\chi^2(t)|}{(8h_{n-1}M)^{1/2}\log(T_{h_n}/h_{n-1})} \leqslant 1+2\varepsilon \quad \text{a.s.}$$

which, by Condition (3.4.12), results in

$$\overline{\lim_{h\to 0}} \sup_{|t|\leqslant T_h} \sup_{0\leqslant s\leqslant h} \frac{|\chi^2(t+s)-\chi^2(t)|}{(8hM)^{1/2}\log(T_h/h)} \leqslant 1 \quad \text{a.s.} \qquad (3.4.13)$$

In order to complete the proof of the theorem, it is enough to show that under Condition (3.4.10)

$$\underline{\lim_{h\to 0}} \sup_{|t|\leqslant T_h} \frac{|\chi^2(t+s)-\chi^2(t)|}{(8hM)^{1/2}\log(T_h/h)} \geqslant 1-\varepsilon \quad \text{a.s.} \qquad (3.4.14)$$

From the condition $\Gamma_2<\infty$, and similarly to (3.4.13) it is easy to see that

$$\overline{\lim_{h\downarrow 0}} \sup_{|t|\leqslant T_h} \frac{|\sum_{k=K}^{\infty}(X_k^2(t+h)-X_k^2(t))|}{(8hM)^{1/2}\log(T_h/h)} \leqslant \varepsilon \quad \text{a.s.} \qquad (3.4.15)$$

provided that $K=K(\varepsilon)$ is large enough. Fixing the $K$, by (3.4.15), we need only to prove

$$\underline{\lim_{h\downarrow 0}} \sup_{|t|\leqslant T_h} \frac{|\sum_{k=1}^{K}(X_k^2(t+h)-X_k^2(t))|}{(8hM)^{1/2}\log(T_h/h)} \geqslant 1-\varepsilon \quad \text{a.s.} \qquad (3.4.16)$$

Define $h_n$ such that $T_{h_{n-1}}/h_n=n$ and put $\xi_l^k=X_k((l+1)h_n)-X_k(lh_n)$, $\eta_l^k=X_k((l+1)h_n)+X_k(lh_n)$. Then

$$\sigma_{ll}^k := E(\xi_l^k)^2 = \frac{2\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_n}),$$

$$\alpha_{ll}^k := E(\eta_l^k)^2 = \frac{2\gamma_k}{\lambda_k}(1+e^{-\lambda_k h_n}),$$

and for $l<r$,

$$\sigma_{lr}^k := E\xi_l^k\xi_r^k = \frac{\gamma_k}{\lambda_k} e^{-\lambda_k(r-l)h_n}(2-e^{-\lambda_k h_n}-e^{\lambda_k h_n}),$$

$$\alpha_{lr}^k := E\eta_l^k\eta_r^k = \frac{\gamma_k}{\lambda_k} e^{-\lambda_k(r-l)h_n}(2+e^{-\lambda_k h_n}+e^{\lambda_k h_n}),$$

$$\tau_{lr}^k := E\xi_l^k\eta_r^k = \frac{\gamma_k}{\lambda_k} e^{-\lambda_k(r-l)h_n}(e^{\lambda_k h_n}-e^{-\lambda_k h_n}),$$

and, clearly, $\tau^k_{rl} = -\tau^k_{lr}$. Let

$$\xi^k_{1,l} = \xi^k_l - \frac{\sigma^k_{1l}}{\sigma^k_{11}}\,\xi^k_1 - \frac{\tau^k_{l1}}{\alpha^k_{11}}\,\eta^k_1, \qquad \eta^k_{1,l} = \eta^k_l - \frac{\tau^k_{1l}}{\sigma^k_{11}}\,\xi^k_1 - \frac{\alpha^k_{1l}}{\alpha^k_{11}}\,\eta^k_1.$$

It is not difficult to check that $(\xi^k_1, \eta^k_1)$ is independent of $(\xi^k_{1,l}, \eta^k_{1,l})$. We write

$$\begin{aligned}\sum_{k=1}^K \xi^k_l \eta^k_l = \sum_{k=1}^K \Big\{ & \xi^k_{1,l}\eta^k_{1,l} + \frac{\sigma^k_{1l}}{\sigma^k_{11}}\,\xi^k_1\eta^k_l - \frac{\sigma^k_{1l}\tau^k_{1l}}{(\sigma^k_{11})^2}\,(\xi^k_1)^2 - \frac{\sigma^k_{1l}\alpha^k_{1l}}{\sigma^k_{11}\alpha^k_{11}}\,\xi^k_1\eta^k_1 \\ & + \frac{\tau^k_{l1}}{\alpha^k_{11}}\,\eta^k_1\eta^k_l - \frac{\tau^k_{1l}}{\sigma^k_{11}} \cdot \frac{\tau^k_{l1}}{\alpha^k_{11}}\,\xi^k_1\eta^k_1 - \frac{\alpha^k_{1l}\tau^k_{l1}}{(\alpha^k_{11})^2}\,(\eta^k_1)^2 \\ & + \frac{\tau^k_{1l}}{\sigma^k_{11}}\,\xi^k_1\xi^k_l + \frac{\alpha^k_{1l}}{\alpha^k_{11}}\,\xi^k_l\xi^k_1 \Big\} \\ =: & \sum_{k=1}^K \xi^k_{1,l}\eta^k_{1,l} + H_l.\end{aligned}$$

Put

$$\lambda^m = \min\{\lambda_k, k \leqslant K\} > 0, \; L = [(\lambda^m h_n)^{-1}\log(T_{h_{n-1}}/h_n)],$$

$$A_n = (1-\varepsilon)(8h_nM)^{1/2}\log(T_{h_{n-1}}/h_n).$$

We have

$$\begin{aligned} & P\Big\{ \max_{|l| \leqslant [T_{h_{n-1}}/h_n]} \Big|\sum_{k=1}^K (X^2_k((l+1)h_n) - X^2_k(lh_n))\Big| \leqslant A_n \Big\} \qquad (3.4.17) \\ & \leqslant P\Big\{ \max_{1 \leqslant j \leqslant T_{h_{n-1}}/(Lh_n)} \Big|\sum_{k=1}^K (X^2_k((jL+1)h_n) - X^2_k(jLh_n))\Big| \leqslant A_n \Big\} \\ & \leqslant P\Big\{ \Big|\sum_{k=1}^K (X^2_k((L+1)h_n) - X^2_k(Lh_n))\Big| \leqslant A_n \Big\} \\ & \quad \cdot P\Big\{ \max_{2 \leqslant j \leqslant T_{h_{n-1}}/(Lh_n)} \Big|\sum_{k=1}^K \xi^k_{1,jL}\eta^k_{1,jL}\Big| \leqslant A_n(1 + (T_{h_{n-1}}/h_n)^{-1}) \Big\} \\ & \quad + P\Big\{ \max_{2 \leqslant j \leqslant T_{h_{n-1}}/(Lh_n)} |H_{jL}| > A_n(T_{h_{n-1}}/h_n)^{-1} \Big\}.\end{aligned}$$

At first, we wish to estimate the last probability. A typical term in $H_{jL}$ is $\sum_{k=1}^K (\sigma^k_{1,jL}/\sigma^k_{11})\xi^k_1\eta^k_{jL}$, which we now proceed to estimate in probability. We put $\overline{\eta^k_{jL}} = \eta^k_{jL} - (\tau^k_{1,jL}/\sigma^k_{11})\xi^k_1$ and note that the latter is independent of $\xi^k_1$. We have

$$E(\overline{\eta}_{jL}^{k})^2=\alpha_{jL,jL}^{2}-(\tau_{1,jL}^{k})^2/\sigma_{11}^{k}.$$

It suffices to estimate only $\sum_{k=1}^{K}(\sigma_{1,jL}^{k}/\sigma_{11}^{k})\xi_1^k\overline{\eta}_{jL}^{k}$. By imitating the proof of the right-hand side inequality of (3.4.4), we obtain

$$P\{|\sum_{k=1}^{K}(\sigma_{1,jL}^{k}/\sigma_{11}^{k})\xi_1^k\overline{\eta}_{jL}^{k}|>\frac{1}{16}A_n(T_{h_{n-1}}/h_n)^{-1}\} \tag{3.4.18}$$
$$\leqslant 2\exp\{-\frac{1}{16}(1-\frac{\varepsilon}{2})A_n(T_{h_{n-1}}/h_n)^{-1}/((\sigma_{1,jL}^{k0})^2(\sigma_{11}^{k0})^{-1}(\alpha_{jL,jL}^{k0}-(\tau_{1,jL}^{k0})^2/\sigma_{11}^{k0}))^{1/2}$$
$$+\frac{1}{\varepsilon}(\sum_{k=1}^{K}(\sigma_{1,jL}^{k})^2(\sigma_{11}^{k})^{-1}(\alpha_{jL,jL}^{k}-(\tau_{1,jL}^{k})^2/\sigma_{11}^{k})^2(\sigma_{1,jL}^{k0})^2(\sigma_{11}^{k0})^{-1}$$
$$\cdot(\alpha_{jL,jL}^{k0}-(\tau_{1,jL}^{k0})^2/\sigma_{11}^{k0}))\},$$

where

$$\begin{aligned}(\sigma_{1,jL}^{k})^2(\sigma_{11}^{k})^{-1}&\leqslant(\sigma_{1L}^{k})^2(\sigma_{11}^{k})^{-1}\\ &\leqslant O((\gamma_k\lambda_k h_n^2(T_{h_{n-1}}/h_n)^{-1})^2/(\gamma_k/\lambda_k))\\ &=O(\gamma_k\lambda_k^3h_n^6T_{h_{n-1}}^{-2})\end{aligned}$$

and

$$\alpha_{jL,jL}^{k}-(\tau_{1,jL}^{k})^2/\sigma_{11}^{k}\sim 2\gamma_k/\lambda_k.$$

Inserting these into (3.4.18) yields

$$P\{|\sum_{k=1}^{K}(\sigma_{1,jL}^{k}/\sigma_{11}^{k})\xi_1^k\overline{\eta}_{jL}^{k}|\geqslant\frac{1}{16}A_n(T_{h_{k-1}}/h_n)^{-1}\}$$
$$\leqslant 2\exp\{-ch_n^{-3/2}\log(T_{h_{n-1}}/h_n)\}\leqslant(T_{h_{n-1}}/h_n)^{-4},$$

provided that $n$ is large enough. Consequently, we have also

$$P\{\max_{2\leqslant j\leqslant T_{h_{n-1}}/(Lh_n)}|\sum_{k=1}^{K}(\sigma_{1,jL}^{k}/\sigma_{11}^{k})\xi_1^k\overline{\eta}_{jL}^{k}|\geqslant\frac{1}{16}A_n(T_{h_{n-1}}/h_n)^{-1}\}$$
$$\leqslant L^{-1}(T_{h_{n-1}}/h_n)^{-3}.$$

For the other terms of $H_{jL}$ we have similar estimations, and thus we arrive at

$$P\{\max_{2\leqslant j\leqslant T_{h_{n-1}}/(Lh_n)}|H_{jL}|\geqslant A_n(T_{h_{n-1}}/h_n)^{-1}\}\leqslant cL^{-1}(T_{h_{n-1}}/h_n)^{-3}.$$

Using a similar procedure for estimating the second probability on the right-hand side of the inequality of (3.4.17), we obtain

$$P\{\max_{2\leqslant j\leqslant T_{hn-1}/(Lh_n)} |\sum_{k=1}^{K} \xi_{1,jL}^k \eta_{1,jL}^k| \leqslant A_n(1+(T_{h_{n-1}}/h_n)^{-1})\} \tag{3.4.19}$$

$$\leqslant P\{\max_{2\leqslant j\leqslant T_{hn-1}/(Lh_n)} |\sum_{k=1}^{K} \xi_{jL}^k \eta_{jL}^k| \leqslant A_n(1+2(T_{h_{n-1}}/h_n)^{-1})\}$$

$$+cL^{-1}(T_{h_{n-1}}/h_n)^{-3}.$$

Inserting the latter upper estimate into (3.4.17) and then repeating the same procedure for estimating the probabilities on the right-hand side of the inequality of (3.4.19), we continue this procedure until we obtain

$$P\{\max_{|l|\leqslant[T_{hn-1}/h_n]} |\sum_{k=1}^{K}(X_k^2((l+1)h_n)-X_k^2(lh_n))|\leqslant A_n\} \tag{3.4.20}$$

$$\leqslant \prod_{j=1}^{[T_{hn-1}/Lh_n]} P\{|\sum_{k=1}^{K}(X_k^2((jL+1)h_n)-X_k^2(jLh_n))| \leqslant A_n(1$$

$$+j(T_{h_{n-1}}/h_n)^{-1})\}+c(LT_{h_{n-1}}/h_n)^{-2}$$

$$\leqslant (P\{|\sum_{k=1}^{K}(X_k^2(h_n)-X_k^2(0))|$$

$$\leqslant (1+\frac{\varepsilon}{3})A_n\})^{[T_{hn-1}/(Lh_n)]}+c(LT_{h_{n-1}}/h_n)^{-2}.$$

Having taken the value of $K=K(\varepsilon)$ large enough, and by using now (3.4.1) and our Condition (3.4.10), the first term on the right-hand side of the last inequality of (3.4.20) does not exceed

$$(P\{|\chi^2(h_n)-\chi^2(0)|\leqslant(1-\frac{2\varepsilon}{3})(8h_nM)^{1/2}\log(T_{h_{n-1}}/h_n)\})^{[T_{hn-1}/(Lh_n)]}$$

$$\leqslant(1-\frac{1}{8\log(T_{h_{n-1}}/h_n)}\exp\{-(1-\frac{\varepsilon}{2})\log(T_{h_{n-1}}/h_n)\})^{[T_{hn-1}/(Lh_n)]}$$

$$\leqslant\exp\{-(T_{h_{n-1}}/h_n)^{\varepsilon/3}L^{-1}\}$$

$$\leqslant\exp\{-T_{h_{n-1}}^{\varepsilon/4}\}\leqslant n^{-2},$$

provided that $n$ is large enough. The rest of this proof of (3.4.16) is similar to that of (3.3.23), and hence we omit these further details.

### 3.4.2 A law of the logarithm type

At first, we give the following large deviation results. Put $m = \max_{j \geq 1} \gamma_j / \lambda_j$.

**Lemma 3.4.2** *Assume $\Gamma_0 < \infty$. Then for any $\varepsilon > 0$ we have*

$$P\{\chi^2(t) \geq 2mv\} \geq \frac{\varepsilon}{6} v^{1/2} \exp\left(-\frac{v}{1-\varepsilon}\right) \tag{3.4.21}$$

*for any $t \geq 0$ and $v > 0$. If, in addition, $\Gamma_2 < \infty$ as well, then there exist $C = C(\varepsilon) > 0$ and $v_0 = v_0(\varepsilon) > 0$ such that for any $T > 0$*

$$P\{\max_{|t| \leq T} \chi^2(t) \geq 2mv\} \leq CT \exp\left(-\frac{v}{1+\varepsilon}\right), \tag{3.4.22}$$

*provided that $v \geq v_0$.*

*Proof* (3.4.22) is an immediate consequence of Theorem 1 of Iscoe and McDonald (1989). We only prove (3.4.21).

Let $m_n = \max_{1 \leq j \leq n} \gamma_j / \lambda_j$, $\tau_n = \sum_{j=1}^{n} \gamma_j / \lambda_j$. At first, we show

$$P\{\sum_{j=1}^{n} X_j^2(t) \geq 2m_n v\} \geq \frac{\varepsilon}{6} v^{1/2} \exp\left(-\frac{v}{1-\varepsilon}\right) \tag{3.4.23}$$

for any $v > 0$. Let $k_0$ be an integer such that $\gamma_{k0} / \lambda_{k0} = m_n$. Put $Y = \sum_{j=1}^{n} X_j^2(t) - X_{k0}^2(t)$. Then $Y$ is independent of $X_{k0}^2(t)$. By the central limit theorem, there exists $n_0$ such that for $n \geq n_0$, $P\{Y \geq EY\} \geq 1/3$. Hence, imitating (3.4.5) and noting that $X_{k0}^2(t)/m_n$ has a $\chi_1^2$-distribution, we have

$$\begin{aligned} &P\{\sum_{j=1}^{n} X_j^2(t) \geq 2m_n v\} \\ &\geq \frac{1}{3} P\{X_{k0}^2(t) \geq 2m_n v\} \\ &= \frac{1}{3} \int_{2v}^{\infty} \frac{1}{2^{1/2}\Gamma(1/2)} y^{-1/2} e^{-y/2} dy \\ &\geq \frac{1}{3 \cdot 2^{1/2}\Gamma(1/2)} \int_{2v}^{2v/(1-\varepsilon)} y^{-1/2} e^{-y/2} dy \end{aligned} \tag{3.4.24}$$

$$\geqslant \frac{\varepsilon}{6} v^{1/2} \exp\left(-\frac{v}{1-\varepsilon}\right).$$

(3.4.23) is proved. By noting $\Gamma_0 < \infty$, it is easy to show (3.4.21) from (3.4.23).

**Theorem 3.4.2** (Lin 1990c) *Assume* $\Gamma_0 < \infty$ *and* $\Gamma_2 < \infty$. *Then we have*

$$\lim_{T \to \infty} \sup_{|t| \leqslant T} \chi^2(t)/(2m \log T) = 1 \quad \text{a.s.} \tag{3.4.25}$$

$$\overline{\lim_{T \to \infty}} \chi^2(T)/(2m \log T) = 1 \quad \text{a.s.} \tag{3.4.26}$$

*Proof* It is easy by using (3.4.22) to prove

$$\overline{\lim_{T \to \infty}} \sup_{|t| \leqslant T} \chi^2(t)/(2m \log T) \leqslant 1 \quad \text{a.s.} \tag{3.4.27}$$

Hence, in order to get (3.4.25), it suffices to show

$$\underline{\lim_{T \to \infty}} \sup_{|t| \leqslant T} \chi^2(t)/(2m \log T) \geqslant 1 - \varepsilon \quad \text{a.s.} \tag{3.4.28}$$

for any $\varepsilon > 0$. By $\Gamma_0 < \infty$ we have $\lim\limits_{k \to \infty} \max\limits_{j \geqslant k} \gamma_j / \lambda_j = 0$. Therefore, similarly to (3.4.27) we can prove that there exists an integer $K = K(\varepsilon)$ such that

$$\overline{\lim_{T \to \infty}} \sup_{|t| \leqslant T} \sum_{k=K}^{\infty} X_k^2(t)/(2m \log T) \leqslant \varepsilon \quad \text{a.s.} \tag{3.4.29}$$

Fixing the value of $K$ by (3.4.29) the statement of (3.4.28) becomes equivalent to

$$\underline{\lim_{T \to \infty}} \sup_{|t| \leqslant T} \sum_{k=1}^{K} X_k^2(t)/(2m \log T) \geqslant 1 - \varepsilon \quad \text{a.s.} \tag{3.4.30}$$

Put $x_{ji} = \sum\limits_{k=1}^{K} X_k^2(j) e^{-2\lambda_k (j-i)}$, $\xi_{ji} = \sum\limits_{k=1}^{K} X_k(j)(X_k(i) - X_k(j) e^{-\lambda_k (j-i)}) e^{-\lambda_k (j-i)}$ for $1 \leqslant i \leqslant j$. And put $n' = [n^{1-\varepsilon/6}]$, $n_i = n - i[n^{\varepsilon/6}]$, $i = 1, \cdots, n'$, $n_0 = n$, $\lambda''(K) = \min\limits_{1 \leqslant k \leqslant K} \lambda_k$. We have for $i \leqslant n_j$ and $k \leqslant K$

$$EX_k^2(n_{j-1}) e^{-2\lambda_k (n_{j-1} - i)} = \frac{\gamma_k}{\lambda_k} e^{-2\lambda_k (n_{j-1} - i)} \leqslant \frac{\gamma_k}{\lambda_k} e^{-\lambda''(K) n^{\varepsilon/6}}. \tag{3.4.31}$$

Moreover, $\sum_{k=1}^{K} X_k^2(j)$ is independent of $\sum_{k=1}^{K} X_k^2(i) - \chi_{ji} - 2\xi_{ji}$ for $i < j$, since $\{X_k(j),\ k=1,\cdots,K\}$ is independent of $\{X_k(i) - X_k(j)e^{-\lambda_k(j-i)},\ k=1,\cdots,K, i=1,\cdots,j-1\}$. Then

$$P\{\sup_{|t|\leqslant n} \sum_{k=1}^{K} X_k^2(t)/(2m\log n) \leqslant 1-\varepsilon\}$$

$$\leqslant P\{\max_{i\leqslant n} \sum_{k=1}^{K} X_k^2(i)/(2m\log n) \leqslant 1-\varepsilon\}$$

$$\leqslant P\{\sum_{k=1}^{K} X_k^2(n)/(2m\log n) \leqslant 1-\varepsilon\} P\{\max_{i\leqslant n_1}(\sum_{k=1}^{K} X_k^2(i) - \chi_{ni} - 2\xi_{ni})$$

$$/(2m\log n) \leqslant 1-\varepsilon + \frac{\varepsilon}{4n}\} + P\{\max_{i\leqslant n_1} \xi_{ni}/(2m\log n) \geqslant \frac{\varepsilon}{8n}\}$$

$$\leqslant P\{\sum_{k=1}^{K} X_k^2(n)/(2m\log n) \leqslant 1-\varepsilon\} P\{\max_{i\leqslant n_1}\sum_{k=1}^{K} X_k^2(i)/(2m\log n)$$

$$\leqslant 1-\varepsilon + \frac{\varepsilon}{2n}\} + P\{\max_{i\leqslant n_1} \chi_{ni}/(2m\log n) \geqslant \frac{\varepsilon}{12n}\}$$

$$+ 2P\{\max_{i\leqslant n_1} \xi_{ni}/(2m\log n) \geqslant \frac{\varepsilon}{12n}\}.$$

Inductively, we have

$$P\{\sup_{|t|\leqslant n} \sum_{k=1}^{K} X_k^2(t)/(2m\log n) \leqslant 1-\varepsilon\} \tag{3.4.32}$$

$$\leqslant \prod_{j=1}^{n'} P\{\sum_{k=1}^{K} X_k^2(n_j)/(2m\log n) \leqslant 1-\frac{\varepsilon}{2}\}$$

$$+\sum_{j=1}^{n'} P\{\max_{i\leqslant n_j} \chi_{n_{j-1}i}/(2m\log n) > \varepsilon/(12n)\}$$

$$+2\sum_{j=1}^{n'} P\{\max_{i\leqslant n_j} \xi_{n_{j-1}i}/(2m\log n) > \varepsilon/(12n)\}$$

$$=: p_{n1} + p_{n2} + p_{n3}.$$

Using (3.4.21) with $\varepsilon/6$ and $\sum_{k=1}^{K} X_k^2(t)$ instead of $\varepsilon$ and $\chi^2(t)$ respectively (recalling (3.4.29)) we have for all large $n$,

$$p_{n1} \leqslant \{1 - \frac{\varepsilon}{54}(\log^{1/2} n)\exp(-\frac{(1-\varepsilon/2)\log n}{1-\varepsilon/6})\}^{n'} \leqslant \exp(-n^{\varepsilon/6}).$$

For $p_{n2}$, using an inequality similar to (3.4.22) and noting (3.4.31), we have

$$P\{\max_{i\leqslant n_j}\chi_{n_{j-1}i}\geqslant 2mx\}\leqslant cn_j\exp\{-\frac{x}{1+\varepsilon}e^{\lambda''(K)n^{\varepsilon/6}}\}$$

$$\leqslant cn\exp\{-\frac{x}{1+\varepsilon}e^{\lambda''(K)n^{\varepsilon/6}}\},$$

provided, say, $x\geqslant n^{-1}$. So

$$p_{n2}\leqslant cn^{2-\varepsilon/6}\exp\{-\frac{\varepsilon\log n}{12(1+\varepsilon)n}e^{\lambda''(K)n^{\varepsilon/6}}\}.$$

Consider $p_{n3}$. For $0\leqslant a\leqslant(\lambda_k/\gamma_k)^{-1}e^{\lambda k(j-i)}(1-e^{-2\lambda k(j-i)})^{-1/2}$, by noting that $X_k(j)$ is independent of $(X_k(i)-X_k(j)e^{-\lambda k(j-i)})e^{-\lambda k(j-i)}$,

$$E\exp\{aX_k(j)(X_k(i)-X_k(j)e^{-\lambda k(j-i)})e^{-\lambda k(j-i)}\}$$

$$=E\{E[\exp\{aX_k(j)(X_k(i)-X_k(j)e^{-\lambda k(j-i)})e^{-\lambda k(j-i)}|X_k(j)]\}$$

$$=E\exp\{\frac{1}{2}a^2X_k^2(j)(\gamma_k/\lambda_k)(1-e^{-2\lambda k(j-i)})e^{-2\lambda k(j-i)}\}$$

$$=(1-a^2(\gamma_k/\lambda_k)^2(1-e^{-2\lambda k(j-i)})e^{-2\lambda k(j-i)})^{-1/2}.$$

Let $a=(1-\varepsilon/2)^{1/2}m^{-1}e^{\lambda''(K)(j-i)}$. Using the inequality (3.4.8) we have for $j-i\geqslant n^{\varepsilon/6}$

$$E\exp(a\xi_{ji})=\prod_{k=1}^{K}(1-a^2(\gamma_k/\lambda_k)^2(1-e^{-2\lambda k(j-i)})e^{-2\lambda k(j-i)})^{-1/2}$$

$$\leqslant\exp\{\frac{1}{2}\cdot\frac{2}{\varepsilon}a^2\sum_{k=1}^{K}(\gamma_k/\lambda_k)^2(1-e^{-2\lambda k(j-i)})e^{-2\lambda k(j-i)}\}$$

$$\leqslant\exp(\Gamma_3/\varepsilon m^2),$$

where $\Gamma_3=\sum_{k=1}^{\infty}\gamma_k^2/\lambda_k^2$ which is finite because of $\Gamma_0<\infty$. Consequently, we have for $j-i\geqslant n^{\varepsilon/6}$ and $x\geqslant n^{-1}$

$$P\{\xi_{ji}>2mx\}\leqslant\exp\{-2a\,mx+\Gamma_3/\varepsilon m^2\}\leqslant\exp\{-xe^{\lambda''(K)n^{\varepsilon/6}}\}$$

for large $n$. Then

$$p_{n3}\leqslant 2\sum_{j=1}^{n'}n_j\exp\{-\frac{\varepsilon\log n}{12n}e^{\lambda''(K)n^{\varepsilon/6}}\}$$

$$\leqslant 2n^{2-\varepsilon/6}\exp\{-\frac{\varepsilon\log n}{12n}e^{\lambda''(K)n^{\varepsilon/6}}\}.$$

Combining the estimations of $p_{n1}$, $p_{n2}$ and $p_{n3}$ we obtain from (3.4.32)

$$P\{\sup_{|t|\leqslant n}\sum_{k=1}^{K} X_k^2(t)/(2m\log n)\leqslant 1-\varepsilon\}=O(\exp(-n^{\varepsilon/6}))$$

as $n\to\infty$, which implies

$$\varliminf_{n\to\infty}\sup_{|t|\leqslant n}\sum_{k=1}^{K} X_k^2(t)/(2m\log(n+1))\geqslant 1-\varepsilon \quad \text{a.s.} \tag{3.4.33}$$

Then we obtain (3.4.30), and further (3.4.28). Thus (3.4.25) is proved.

We can get (3.4.26) from (3.4.25) along the lines of the proof of (3.3.30) from (3.3.29), and hence the details are omitted.

## 3.5 Two-Parameter Gaussian Process with Kernel

The two-parameter Gaussian process $X(t, n)$ studied in Section 3.2 can be rewritten as

$$X(t,n)=\sum_{k=1}^{n} X_k(t)=\sum_{k=1}^{n}\int_{-\infty}^{t}\exp(-\lambda_k(t-s))(2\gamma_k)^{1/2}dW_k(s), \tag{3.5.1}$$

which, in turn, leads to a study the two-parameter Gaussian process

$$X(t,v)=\int_0^v\int_{-\infty}^{t}\exp(-\lambda(y)(t-x))(2\gamma(y))^{1/2}dW(x,y), \tag{3.5.2}$$

where $\gamma(y)$ and $\lambda(y)$ are assumed to be positive continuous functions on $[0,\infty)$, and $\{W(x,y);\ -\infty<x<\infty,\ 0\leqslant y<\infty\}$ is a standard two-parameter Wiener process (see Csörgő and Lin 1990a). This brings us to study the two-parameter Gaussian process $\{X(t,v); t\in R^+, t\in R\}$ of the form

$$X(t,v)=\int_0^{\infty}\int_{-\infty}^{\infty}\Gamma(t,v,x,y)dW(x,y), \tag{3.5.3}$$

where the kernel function $\Gamma(t, v, x, y)$ is assumed to be square integrable in $(x, y)$ on $R^+\times R$. It is clear that $X(t, v)$ is a Gaussian process with a mean zero and covariance function

$$\mathrm{Cov}(X(t,v),\ X(s,u))=\int_0^{\infty}\int_{-\infty}^{\infty}\Gamma(t,v,x,y)\Gamma(s,u,x,y)\,dxdy. \tag{3.5.4}$$

Put

$$H_1^2(t,s,v)=E\,(X(t+s,v)-X(t,v))^2,$$

$$X(R(t,s,v,u))=X(t+s,v+u)-X(t,v+u)-X(t+s,v)+X(t,v),$$

$$H_2^2(t,s,v,u)=E(X(R(t,s,v,u)))^2.$$

It is easy to see that

$$H_1^2(t,s,v)=\int_0^{\infty}\int_{-\infty}^{\infty}(\Gamma(t+s,v,x,y)-\Gamma(t,v,x,y))^2dxdy, \tag{3.5.5}$$

$$H_2^2(t,s,v,u)=\int_0^{\infty}\int_{-\infty}^{\infty}(\Gamma(t+s,v+u,x,y)-\Gamma(t,v+u,x,y) \tag{3.5.6}$$

$$-\Gamma(t+s,v,x,y)+\Gamma(t,v,x,y))^2dxdy.$$

The following examples are immediate.

*Example* 1 If $\Gamma(t,v,x,y)=I_{(-\infty,t]\times[0,v]}(x,y)$, $-\infty<t<\infty$, $0\leqslant v<\infty$, then

$$X(t,v)=W(t,v),$$

$$H_1^2(t,s,v)=sv,\ 0\leqslant s<\infty,$$

$$H_2^2(t,s,v,u)=su,\ 0\leqslant s,u<\infty.$$

*Example* 2 If $\Gamma(t,v,x,y)=I_{[0,t]\times[0,v]}(x,y)-tI_{[0,1]\times[0,v]}(x,y)$, $0\leqslant t\leqslant 1, 0\leqslant v<\infty$, then

$$X(t,v)=W(t,v)-tW(l,v)$$

is a Kiefer process (cf. Section 1.15 in Csörgő-Révész 1981),

$$H_1^2(t,s,v)=s(1-s)v, \qquad 0\leqslant s\leqslant 1,$$

$$H_2^2(t,s,v,u)=s(1-s)u, \qquad 0\leqslant s\leqslant 1, 0\leqslant u<\infty.$$

*Example* 3 If, with $-\infty<t<\infty$, $0<v<\infty$,

$$\Gamma(t, v, x, y) = I_{(-\infty, t] \times (0, v]}(x, y) \exp(-\lambda(y)(t-x))(2\gamma(y))^{1/2},$$

where $\lambda(y)$ and $\gamma(y)$ are positive continuous functions on $(0, \infty)$, then $X(t, v)$ is the two-parameter Gaussian process of (3.5.2) with

$$H_1^2(t, s, v) = 2\int_0^v \frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)s))dx,$$

$$H_2^2(t, s, v, u) = 2\int_v^{v+u} \frac{\gamma(x)}{\lambda(x)}(1-\exp(-\lambda(x)s))dx.$$

In this section, we shall establish the moduli of continuity for the process $X(t, v)$ defined by (3.5.3). For the sake of simplicity, only some particular cases are considered. Especially, we assume that $X(t, v)$ is weakly stationary in $t$. Consequently we can write

$$H_1(s, v) = H_1(t, s, v) \text{ and } H_2(s, v, u) = H_2(t, s, v, u).$$

The general conclusions for *the* moduli of continuity and the large increment results can be found in Csörgő, Lin and Shao (1991).

Moreover, in this section we throughout assume that $H_1(s, v)$ is non-decreasing on $s$ and $H_2(s, v, u)$ is non-decreasing on $s$ and $u$ and that $H_1(s, v)$ and $H_2(s, v, u)$ both are continuous on their every argument.

### 3.5.1 Large deviations

The following lemma is a generalization of the Slepian Lemma.

**Lemma 3.5.1** (Gordon 1985) *Let* $\{X_{ij}\}_I$, $\{Y_{ij}\}_I$, $I = \{(i, j): 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ *be two collections of centered Gaussian variables satisfying the following conditions:*

$$EX_{ij}^2 = EY_{ij}^2 \qquad (i, j) \in I,$$

$$EX_{ij}X_{ik} \leqslant EY_{ij}Y_{ik}, \qquad (i, j), (i, k) \in I,$$

$$EX_{ij}X_{lk} \geqslant EY_{ij}Y_{lk}, \qquad (i, j), (l, k) \in I, i \neq l.$$

*Then, for all* $\lambda_{ij} > 0$

$$P\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(X_{ij}>\lambda_{ij})\}\geqslant P\{\bigcap_{i=1}^{n}\bigcup_{j=1}^{m}(Y_{ij}>\lambda_{ij})\}. \tag{3.5.7}$$

**Lemma 3.5.2** *Let $A\subset R^+$, $s_0>0$. Suppose that*

$$E(X(t+s,v)-X(t,v))(X(t+s,u)-X(t,u)) \tag{3.5.8}$$
$$\geqslant E(X(t+s,u)-X(t,u))^2$$

*for any $t$, $s$, and $v\geqslant u$ and that there exist $c_0>0$ and $\alpha>0$ such that*

$$H_1(s,T)/s^{\alpha}\leqslant c_0H_1(s_1,T)/s_1^{\alpha} \tag{3.5.9}$$

*for any $T\in A$, $0\leqslant s\leqslant s_1\leqslant s_0$. Then for any $0<\varepsilon<1$ there exists $c=c(\varepsilon)>0$ depending only on $c_0$, $\alpha$ and $\varepsilon$ such that*

$$P\{\sup_{T\in A}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\frac{|X(t+s,T)-X(t,T)|}{xH_1(s_0,T^*)}\geqslant 1+\varepsilon\} \tag{3.5.10}$$
$$\leqslant cs_0^{-1}\exp(-x^2/2)$$

*for any $x\geqslant 1$, where $T^*=\sup\{T:T\in A\}$.*

*Proof* Let $Z(T)$ be an independent increment process with $Z(T)\stackrel{\mathscr{D}}{=}X(t+s,T)-X(t,T)$. Then $EZ^2(T)=H_1^2(s,T)$ and

$$EZ(T)Z(T')=H_1^2(s,T')$$
$$\leqslant E(X(t+s,T)-X(t,T))(X(t+s,T')-X(t,T'))$$

for $T\geqslant T'$ by (3.5.8). Furthermore, (3.5.8) implies that $H_1(s,T)$ is non-decreasing on $T$. Then, by the Slepian lemma, we have

$$P\{\sup_{T\in A}\frac{|X(t+s,T)-X(t,T)|}{xH_1(s,T^*)}\geqslant 1\} \tag{3.5.11}$$
$$\leqslant P\{\sup_{T\in A}\frac{X(t+s,T)-X(t,T)}{xH_1(s,T^*)}\geqslant 1\}+P\{\sup_{T\in A}\frac{-(X(t+s,T)-X(t,T))}{xH_1(s,T^*)}\geqslant 1\}$$
$$\leqslant P\{\sup_{T\in A}\frac{Z(T)}{xH_1(s,T^*)}\geqslant 1\}+P\{\sup_{T\in A}\frac{-Z(T)}{xH_1(s,T^*)}\geqslant 1\}$$
$$\leqslant 2P\{\sup_{T\in A}\frac{|Z(T)|}{xH_1(s,T^*)}\geqslant 1\}\leqslant 4\exp(-x^2/2).$$

Let $t_{k+j}=([t2^{2^{k+j}}/s_0]+1)s_0/2^{2^{k+j}}, j=0,1,\cdots$ . By Condition (3.5.9), for any $\varepsilon>0$, there exists an $M>0$ such that

$$\int_M^{\infty} H_1(s_0e^{-z^2}, T)dz<\varepsilon .$$

Therefore, according to a result of Fernique (cf. Corollary 3.2.5 in Jain and Marcus 1978), $X(t, T)$ is a.s. continuous on $t$ for every fixed $T$. Hence we can write

$$\begin{aligned}|X(t+s,T)-X(t,T)| &\leqslant |X((t+s)_k, T)-X(t_k, T)| \qquad (3.5.12)\\ &+\sum_{j=0}^{\infty}|X((t+s)_{k+j+1}, T)-X((t+s)_{k+j}, T)|\\ &+\sum_{j=0}^{\infty}|X(t_{k+j+1}, T)-X(t_{k+j}, T)|.\end{aligned}$$

Put $K=2^{2^k}$ . By the definitions of $H_1(s, T)$ and $t_{k+j}$ and Condition (3.5.9), it is clear that for $k$ large enough and $0\leqslant s\leqslant s_0$ ,

$$\begin{aligned}&H_1((t+s)_k-t_k, T)\leqslant H_1(s_0, T)+2H_1(s_0/K, T) \qquad (3.5.13)\\ &\leqslant(1+2c_0K^{-\alpha})H_1(s_0, T)\leqslant(1+\varepsilon/2)H_1(s_0, T),\end{aligned}$$

$$\begin{aligned}&H_1((t+s)_{k+j}-(t+s)_{k+j+1}, T)\leqslant 2H_1(s_0/2^{2^{k+j}}, T)\\ &\leqslant 2c_0 2^{-\alpha 2^{k+j}}H_1(s_0, T), \qquad (3.5.14)\end{aligned}$$

as well as

$$H_1(t_{k+j}-t_{k+j+1}, T)\leqslant 2c_0 2^{-\alpha 2^{k+j}}H_1(s_0, T). \qquad (3.5.15)$$

From (3.5.13) and (3.5.11), we obtain

$$\begin{aligned}&P\{\sup_{T\in A}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\frac{|X((t+s)_k, T)-X(t_k, T)|}{(1+\varepsilon/2)xH_1(s_0, T^*)}\geqslant 1\} \qquad (3.5.16)\\ &\leqslant c2^{2^{k+1}}s_0^{-1}\exp(-x^2/2).\end{aligned}$$

Similarly, by (3.5.14), (3.5.15) and (3.5.11), we have for any $x_j>0$

$$\begin{aligned}&P\{\sup_{T\in A}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\frac{|X((t+s)_{k+j+1}, T)-X((t+s)_{k+j}, T)|}{2c_0x_j2^{-\alpha 2^{k+j}}H_1(s_0, T^*)}\geqslant 1\}\\ &\leqslant c2^{2^{k+j+1}}s_0^{-1}\exp(-x_j^2/2), \qquad (3.5.17)\end{aligned}$$

and

$$P\{\sup_{T\in A}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\frac{|X(t_{k+j+1},T)-X(t_{k+j},T)|}{2c_0x_j2^{-\alpha 2^{k+j}}H_1(s_0,T^*)}\geqslant 1\}\quad (3.5.18)$$

$$\leqslant c2^{2^{k+j+1}}s_0^{-1}\exp(-x_j^2/2).$$

From (3.5.12), (3.5.16)—(3.5.18) we conclude

$$P\{\sup_{T\in A}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\frac{|X(t+s,T)-X(t,T)|}{((1+\varepsilon/2)x+4c_0\sum_{j=0}^{\infty}x_j2^{-\alpha 2^{k+j}})H_1(s_0,T^*)}\geqslant 1\}$$

$$\leqslant cs_0^{-1}\{2^{2^{k+1}}\exp(-\frac{x^2}{2})+\sum_{j=0}^{\infty}2^{2^{k+j+1}}\exp(-\frac{x_j^2}{2})\}.\qquad (3.5.19)$$

Let $x_j^2=x^2+2^{k+j+2}$. It follows that

$$\sum_{j=0}^{\infty}2^{2^{k+j+1}}e^{-x_j^2/2}=e^{-x^2/2}\sum_{j=0}^{\infty}(2/e)^{2^{k+j+1}}\leqslant 2e^{-x^2/2},\qquad (3.5.20)$$

$$4c_0\sum_{j=0}^{\infty}x_j2^{-\alpha 2^{k+j}}\qquad (3.5.21)$$

$$\leqslant 4c_0x\sum_{j=0}^{\infty}2^{-\alpha 2^{k+j}}+4c_0\sum_{j=0}^{\infty}2^{(k+j+2)/2-\alpha 2^{k+j}}<\varepsilon x/2$$

provided that $k$ is large enough. Now combining (3.5.20), (3.5.21) with (3.5.19) yields (3.5.10).

**Lemma 3.5.3** *Let $A\subset R^+$, $s_0$, $u_0>0$. Suppose that*

$$EX(R(t,s,v',b-v'))X(R(t,s,v,b-v))\qquad (3.5.22)$$

$$\geqslant E(X(R(t,s,v,b-v)))^2$$

*for any $t,s,0<v'\leqslant v\leqslant b$ and that there exist $c_0>0$ and $\alpha>0$ such that*

$$H_2(s,v,u)/s^\alpha\leqslant c_0H_2(s_1,v,u)/s_1^\alpha\qquad (3.5.23)$$

*for $0\leqslant s\leqslant s_1\leqslant s_0$, $0\leqslant v\leqslant 1+u_0$, $0\leqslant u\leqslant 2u_0$. Moreover, suppose that for any $\varepsilon>0$ there exists $\delta_0>0$ such that for $0<\delta\leqslant\delta_0$,*

$$\sup_{0\leqslant s\leqslant s_0}\sup_{0\leqslant v\leqslant 1+u_0}\sup_{-\delta u_0\leqslant u\leqslant u_0}\frac{H_2(s_0,v+u,\delta u_0)+H_2(\delta s_0,v+u,u_0)}{H_2(s_0,v,u_0)}\leqslant\varepsilon.$$

$$(3.5.24)$$

*Then for any* $0<\varepsilon<1$ *there exists* $C=C(\varepsilon)>0$ *depending only on* $c_0$, $\alpha$ *and* $\varepsilon$ *such that*

$$P\{\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\sup_{0\leqslant u\leqslant u_0}\frac{|X(R(t,s,v,u))|}{xH_2(s_0,v,u_0)}\geqslant 1\}\tag{3.5.25}$$

$$\leqslant Cs_0^{-1}u_0^{-1}\exp(-x^2/2)$$

*for any* $x\geqslant 1$.

*Proof* Define

$$t_{k+j}=([t2^{2^{k+j}}/s_0]+1)s_0/2^{2^{k+j}},\ v'_{k+j}=([v2^{2^{k+j}}/u_0]+1)u_0/2^{2^{k+j}}.$$

Noting that $X(R(t,s,v,u))$ is defined with a rectangle $[t,t+s]\times[v,v+u]$, we have

$$|X(R(t,s,v,u))|\leqslant|X(R(t_k,(t+s)_k-t_k,v'_k,(v+u)'_k-v'_k))|\tag{3.5.26}$$
$$+\sum_{j=0}^{\infty}|X(R((t+s)_{k+j+1},(t+s)_{k+j}-(t+s)_{k+j+1},v'_k,(v+u)'_k-v'_k))|$$
$$+\sum_{j=0}^{\infty}|X(R(t_{k+j+1},t_{k+j}-t_{k+j+1},v'_k,(v+u)'_k-v'_k))|$$
$$+|X(R(t,s,v,v'_k-v))|+|X(R(t,s,v+u,(v+u)'_k-(v+u)))|.$$

From (3.5.22), it follows that

$$H_2(s,v',u')\leqslant H_2(s,v,u)\tag{3.5.27}$$

for any $v\leqslant v'$, $v+u\geqslant v'+u'$. Using (3.5.24) and (3.5.27), we get

$$H_2((t+s)_k-t_k,v'_k,(v+u)'_k-v'_k)$$
$$\leqslant H_2(s,v,u)+H_2((t+s)_k-(t+s),v'_k,(v+u)'_k-v'_k)$$
$$+H_2(t_k-t,v'_k,(v+u)'_k-v'_k)+H_2(s,v,v'_k-v)$$
$$+H_2(s,v+u,(v+u)'_k-(v+u))$$
$$\leqslant H_2(s,v,u)+2H_2(s_0/K,v'_k,u_0(1+1/K))$$
$$+H_2(s,v,u_0/K)+H_2(s,v+u,u_0/K)$$
$$\leqslant(1+\varepsilon/4)H_2(s,v,u)$$

for large $K$. Using (3.5.23), (3.5.24) and (3.5.27), similar to (3.5.14) and (3.5.15), we get

$$H_2((t+s)_{k+j}-(t+s)_{k+j+1}, v_k', (v+u)_k'-v_k')$$

$$\leqslant 2H_2(s_0/2^{2^{k+j}}, v, u_0(1+1/K))$$

$$\leqslant 2c_0 2^{-\alpha 2^{k+j}} H_2(s_0, v, u_0(1+1/K))$$

$$\leqslant 3c_0 2^{-\alpha 2^{k+j}} H_2(s_0, v, u_0),$$

$$H_2(t_{k+j}-t_{k+j+1}, v_k', (v+u)_k'-v_k')$$

$$\leqslant 3c_0 2^{-\alpha 2^{k+j}} H_2(s_0, v, u_0).$$

Then, for any $x_j>0$,

$$P\{\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\sup_{0\leqslant u\leqslant u_0}\frac{|X(R(t_k, (t+s)_k-t_k, v_k', (v+u)_k'-v_k'))|}{(1+\varepsilon/4)xH_2(s, v, u)}\geqslant 1\}$$

$$\leqslant 4\cdot 2^{2^{k+2}}s_0^{-1}u_0^{-1}\exp(-x^2/2), \tag{3.5.28}$$

$$P\{\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\sup_{0\leqslant u\leqslant u_0} \tag{3.5.29}$$

$$\frac{|X(R((t+s)_{k+j+1}, (t+s)_{k+j}-(t+s)_{k+j+1}, v_k', (v+u)_k'-v_k'))|}{3c_0x_j2^{-\alpha 2^{k+j}}H_2(s_0, v, u_0)}\geqslant 1\}$$

$$\leqslant 4\cdot 2^{2^{k+1}+2^{k+j+1}}s_0^{-1}u_0^{-1}\exp(-x_j^2/2),$$

$$P\{\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\sup_{0\leqslant u\leqslant u_0}\frac{|X(R(t_{k+j+1}, t_{k+j}-t_{k+j+1}, v_k', (v+u)_k'-v_k'))|}{3c_0x_j2^{-\alpha 2^{k+j}}H_2(s_0, v, u_0)}\geqslant 1\}$$

$$\leqslant 4.2^{2^{k+1}}+2^{k+j+1}s_0^{-1}u_0^{-1}\exp(-x_j^2/2). \tag{3.5.30}$$

Let $x_j^2=x^2+2^{k+j+2}$. Similar to (3.5.20) and (3.5.21),

$$\sum_{j=0}^{\infty}2^{2^{k+1}}+2^{k+j+1}e^{-x_j^2/2}\leqslant 2K^2e^{-x^2/2}, \tag{3.5.31}$$

$$6c_0\sum_{j=0}^{\infty}x_j2^{-\alpha 2^{k+j}}\leqslant \varepsilon x/4. \tag{3.5.32}$$

Now deal with the last but one term of (3.5.26). Puting $d_i=(i+1)u_0/K$, we have for any $y>0$

$$P\{\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant s_0}|X(R(t, s, v, v_k'-v))|\geqslant y\} \tag{3.5.33}$$

$$\leqslant P\{ \max_{0\leqslant i\leqslant K/u_0} \sup_{d_{i-1}\leqslant v\leqslant d_i} \sup_{|t|\leqslant 1} \sup_{0\leqslant s\leqslant s_0} |X(R(t,s,v,v_k'-v))|\geqslant y \}$$

$$\leqslant \sum_{i=0}^{[K/u_0]} P\{ \sup_{d_{i-1}\leqslant v\leqslant d_i} \sup_{|t|\leqslant 1} \sup_{0\leqslant s\leqslant s_0} |X(R(t,s,v,d_i-v))|\geqslant y \}.$$

Let $Z(\cdot)$ be an independent increment process with $Z(d_i-v)\overset{\mathscr{D}}{=}X(R(t,s,v,d_i-v))$ for $d_{i-1}\leqslant v<d_i$. Then, for any $v\geqslant v'$

$$EZ(d_i-v)Z(d_i-v')=EZ^2(d_i-v)=EX^2(R(t,s,v,d_i-v))$$

$$\leqslant EX(R(t,s,v,d_i-v))X(R(t,s,v',d_i-v')),$$

where the last inequality is due to (3.5.22). By (3.5.27), we find that

$$H_2(s,v-u_0/K,2u_0/K)\geqslant H_2(s,d_{i-1},u_0/K)$$

for any $d_{i-1}\leqslant v<d_i$. Therefore, using the Slepian Lemma, we obtain

$$P\{ \sup_{d_{i-1}\leqslant v<d_i} \frac{|X(R(t,s,v,d_i-v))|}{xH_2(s,v-u_0/K,2u_0/K)}\geqslant 1\} \tag{3.5.34}$$

$$\leqslant 2P\{ \sup_{d_{i-1}\leqslant v<d_i} \frac{|Z(d_i-v)|}{xH_2(s,v-u_0/K,2u_0/K)}\geqslant 1\}$$

$$\leqslant 2P\{ \frac{|Z(d_i-d_{i-1})|}{xH_2(s,d_{i-1},u_0/K)}\geqslant 1 \}\leqslant 4\exp(-x^2/2).$$

By noting (3.5.24), (3.5.34) implies

$$P\{ \sup_{d_{i-1}\leqslant v<d_i} \frac{|X(R(t,s,v,d_i-v))|}{(\varepsilon/16)xH_2(s,v,u_0)}\geqslant 1 \}\leqslant 4\exp(-x^2/2). \tag{3.5.35}$$

Proceeding along the lines of the proof of (3.5.10), we can get from (3.5.35)

$$P\{ \sup_{d_{i-1}\leqslant v<d_i} \sup_{|t|\leqslant 1} \sup_{0\leqslant s\leqslant s_0} \frac{|X(R(t,s,v,d_i-v))|}{(\varepsilon/8)xH_2(s,v,u_0)}\geqslant 1 \} \tag{3.5.36}$$

$$\leqslant 8\cdot 2^{2^{k+1}}s_0^{-1}\exp(-x^2/2).$$

Combining (3.5.33) with (3.5.36) yields

$$P\{ \sup_{|t|\leqslant 1} \sup_{0\leqslant v\leqslant 1} \sup_{0\leqslant s\leqslant s_0} \frac{|X(R(t,s,v,v_k'-v))|}{(\varepsilon/8)xH_2(s,v,u_0)}\geqslant 1 \} \tag{3.5.37}$$

$$\leqslant 16\cdot 2^{2^{k+2}}s_0^{-1}u_0^{-1}\exp(-x^2/2).$$

Similarly, for the last term of (3.5.26), we have

$$P\{\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant s_0}\frac{|X(R(t,s,v+u,(v+u)_k'-(v+u)))|}{(\varepsilon/8)xH_2(s,v,u_0)}\geqslant 1\}$$

$$\leqslant 16\cdot 2^{2^{k+2}}s_0^{-1}u_0^{-1}\exp(-x^2/2). \tag{3.5.38}$$

(3.5.25) now follows from (3.5.26), (3.5.28)—(3.5.32), (3.5.37) and (3.5.38). This completes the proof of Lemma 3.5.3.

*Remark* 3.5.1 It is easy to see from the proof of Lemmas 3.5.2 and 3.5.3 that "$\sup_{|t|\leqslant 1}$" and "$\sup_{0\leqslant v\leqslant 1}$" can be rewritten by "$\sup_{|t|\leqslant M}$" and "$\sup_{0\leqslant v\leqslant M}$", where $M>0$ is a constant. In this case, $C=C(\varepsilon)$ in (3.5.10) or (3.5.25) ought to be replaced by $C=C(\varepsilon, M)$.

### 3.5.2 Path properties

Applying the large deviation results of $X(t,v)$, we establish its moduli of continuity. Let $a_T$ and $b_T$ be non-negative continuous functions tending to zero as $T\to\infty$.

**Theorem 3.5.1** (Csörgő, Lin, Shao 1991) *Suppose that conditions* (3.5.8) *and* (3.5.9) *are satisfied, Moreover suppose that*

$$E(X((i+1)s,v)-X(is,v))(X((j+1)s,u)-X(js,u))\leqslant 0, \tag{3.5.39}$$

$$\log\log(H_1(a_T,T)+H_1^{-1}(a_T,T))=o(\log(1/a_T)) \text{ as } T\to\infty \tag{3.5.40}$$

*Then we have*

$$\lim_{T\to\infty}\sup_{|t|\leqslant 1}\frac{|X(t+a_T,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}}=1 \quad \text{a.s.} \tag{3.5.41}$$

$$\lim_{T\to\infty}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\frac{|X(t+s,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}}=1 \quad \text{a.s.} \tag{3.5.42}$$

*Proof* At first, we prove

$$\overline{\lim_{T\to\infty}}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\frac{|X(t+s,T)-X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}}\leqslant 1 \quad \text{a.s.} \tag{3.5.43}$$

Let $\theta > 1$. Define $A_{kj} = \{ T : \theta^{-(j+1)} < a_T \leqslant \theta^{-j}, \theta^k \leqslant H_1(\theta^{-j}, T) \leqslant \theta^{k+1} \}$, $j = 0, 1, \cdots, k = \cdots, -1, 0, 1, \cdots$, $T^*_{kj} = \sup\{ T : T \in A_{kj} \}$ and $T'_{kj} = \inf\{ T : T \in A_{kj} \}$. From Condition (3.5.40), for $0 < \varepsilon < 1/2$, $A_{kj} = \emptyset$ if $|k| \geqslant \theta^{\varepsilon j}$ when $j$ is large enough. Using this fact, noting Condition (3.5.9) and taking $\theta$ close to one enough, we have

$$\varlimsup_{T \to \infty} \sup_{|t|<1} \sup_{0 \leqslant s \leqslant a_T} \frac{|X(t+s, T) - X(t, T)|}{H_1(a_T, T)(2\log(1/a_T))^{1/2}} \tag{3.5.44}$$

$$\leqslant \varlimsup_{j \to \infty} \sup_{|k| \leqslant \theta^{\varepsilon j}} \sup_{T \in A_{kj}} \sup_{|t| \leqslant 1} \sup_{0 \leqslant s \leqslant a_T} \frac{|X(t+s, T) - X(t, T)|}{H_1(a_T, T)(2\log(1/a_T))^{1/2}}$$

$$\leqslant \varlimsup_{j \to \infty} \max_{|k| \leqslant \theta^{\varepsilon j}} \sup_{T \in A_{kj}} \sup_{|t| \leqslant 1} \sup_{0 \leqslant s \leqslant \theta^{-j}} \frac{(1+\varepsilon)|X(t+s, T) - X(t, T)|}{\theta^k (2\log \theta^j)^{1/2}}$$

$$\leqslant \varlimsup_{j \to \infty} \max_{|k| \leqslant \theta^{\varepsilon j}} \sup_{T \in A_{kj}} \sup_{|t| \leqslant 1} \sup_{0 \leqslant s \leqslant \theta^{-j}} \frac{\theta(1+\varepsilon)|X(t+s, T) - X(t, T)|}{H_1(\theta^{-j}, T^*_{kj})(2\log \theta^j)^{1/2}}.$$

By Lemma 3.5.2 and (3.5.9) again, we get

$$P\{ \max_{|k| \leqslant \theta^{\varepsilon j}} \sup_{T \in A_{kj}} \sup_{|t| \leqslant 1} \sup_{0 \leqslant s \leqslant \theta^{-j}} \frac{|X(t+s, T) - X(t, T)|}{H_1(\theta^{-j}, T^*_{kj})(2\log \theta^j)^{1/2}} \geqslant 1+\varepsilon \} \tag{3.5.45}$$

$$\leqslant \sum_{|k| \leqslant \theta^{\varepsilon j}} P\{ \sup_{T \in A_{kj}} \sup_{|t| \leqslant 1} \sup_{0 \leqslant s \leqslant \theta^{-j}} \frac{|X(t+s, T) - X(t, T)|}{H_1(\theta^{-j}, T^*_{kj})(2\log \theta^j)^{1/2}} \geqslant 1+\varepsilon \}$$

$$\leqslant C\theta^{\varepsilon j + j} \exp\{ -(1+\varepsilon)^2 \log \theta^j \} \leqslant c\theta^{-\varepsilon j}.$$

(3.5.43) follows from (3.5.44), (3.5.45) and the Borel-Cantelli Lemma.

Next, we prove

$$\varliminf_{T \to \infty} \sup_{|t| \leqslant 1} \frac{|X(t+a_T, T) - X(t, T)|}{H_1(a_T, T)(2\log(1/a_T))^{1/2}} \geqslant 1 \quad \text{a.s.} \tag{3.5.46}$$

which together with (3.5.43) will imply the conclusions of Theorem 3.5.1. Note that

$$\varliminf_{T \to \infty} \sup_{|t| \leqslant 1} \frac{|X(t+a_T, T) - X(t, T)|}{H_1(a_T, T)(2\log(1/a_T))^{1/2}} \tag{3.5.47}$$

$$\geqslant \varliminf_{j \to \infty} \min_{|k| \leqslant \theta^{\varepsilon j}} \inf_{T \in A_{kj}} \sup_{0 \leqslant t \leqslant 1} \frac{|X(t+a_T, T) - X(t, T)|}{H_1(a_T, T)(2\log(1/a_T))^{1/2}}$$

$$\geqslant \varliminf_{j \to \infty} \min_{|k| \leqslant \theta^{\varepsilon j}} \inf_{T \in A_{kj}} \sup_{0 \leqslant t \leqslant 1} \frac{|X(t+\theta^{-j}, T) - X(t, T)|}{H_1(\theta^{-j}, T)(2\log \theta^{j+1})^{1/2}}$$

$$-\overline{\lim_{j\to\infty}} \max_{|k|\leq \theta^{\varepsilon j}} \sup_{T\in A_{kj}} \sup_{0\leq t\leq 1} \frac{|X(t+\theta^{-j},T)-X(t+a_T,T)|}{H_1(\theta^{-j},T)(2\log\theta^j)^{1/2}}$$

$$\geq \underline{\lim_{j\to\infty}} \min_{|k|\leq \theta^{\varepsilon j}} \inf_{T\in A_{kj}} \max_{0\leq i\leq \theta^j} \frac{|X((i+1)\theta^{-j},T)-X(i\theta^{-j},T)|}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}}$$

$$-\overline{\lim_{j\to\infty}} \max_{|k|\leq \theta^{\varepsilon j}} \sup_{T\in A_{kj}} \sup_{a_T\leq t\leq a_T+1} \sup_{0\leq s\leq (\theta-1)\theta^{-j-1}} \frac{|X(t+s,T)-X(t,T)|}{H_1(\theta^{-j},T)(2\log\theta^j)^{1/2}}.$$

Along the lines of the proof of (3.5.43) and noting Remark 3.5.1, we have by (3.5.9)

$$\overline{\lim_{j\to\infty}} \max_{|k|\leq \theta^{\varepsilon j}} \sup_{T\in A_{kj}} \sup_{a_T\leq t\leq a_T+1} \sup_{0\leq s\leq (\theta-1)\theta^{-j-1}} \frac{|X(t+s,T)-X(t,T)|}{H_1(\theta^{-j},T)(2\log\theta^j)^{1/2}}$$

$$\leq \overline{\lim_{j\to\infty}} \max_{|k|\leq \theta^{\varepsilon j}} \sup_{T\in A_{kj}} \sup_{a_T\leq t\leq a_T+1} \sup_{0\leq s\leq (\theta-1)\theta^{-j-1}} \frac{\varepsilon|X(t+s,T)-X(t,T)|}{H_1((\theta-1)\theta^{-j-1},T)(2\log\theta^j)^{1/2}}$$

$$\leq \varepsilon \quad \text{a.s.} \tag{3.5.48}$$

Let $Y(i,T)=X((i+1)\theta^{-j},T)-X(i\theta^{-j},T)$ and $Z(i,T)$ be a two-parameter Gaussian process which is an independent increment process for fixed $i$ with $Z(i,T)\overset{\mathscr{D}}{=}Y(i,T)$ and

$$EZ(i,T)Z(j,T')=EY(i,T)Y(j,T')$$

for $i\neq j$. Then, by (3.5.8) we have

$$EY(i,T)Y(i,T')\geq EY^2(i,T\wedge T')=EZ(i,T)Z(i,T').$$

Hence, we can use Lemma 3.5.1 and obtain

$$P\{\inf_{T\in A_{kj}} \max_{0\leq i\leq \theta^j} \frac{Y(i,T)}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}} \leq \frac{1}{(1+\varepsilon)^2}\} \tag{3.5.49}$$

$$=1-P\{\bigcap_{T\in A_{kj}} \bigcup_{0\leq i\leq \theta^j} (\frac{Y(i,T)}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}} \geq \frac{1}{(1+\varepsilon)^2})\}$$

$$\leq 1-P\{\bigcap_{T\in A_{kj}} \bigcup_{0\leq i\leq \theta^j} (\frac{Z(i,T)}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}} \geq \frac{1}{(1+\varepsilon)^2})\}$$

$$=P\{\bigcup_{T\in A_{kj}} \bigcap_{0\leq i\leq \theta^j} (\frac{Z(i,T)}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}} \leq \frac{1}{(1+\varepsilon)^2})\}.$$

Consequently,

$$P\{\min_{|k|\leqslant\theta^{\varepsilon j}}\inf_{T\in A_{kj}}\max_{0\leqslant i\leqslant\theta^j}\frac{Y(i,T)}{H_1(\theta^{-j},T)(2\log\theta^{j+1})^{1/2}}\leqslant\frac{1}{(1+\varepsilon)^2}\}\tag{3.5.50}$$

$$\leqslant\sum_{|k|\leqslant\theta^{\varepsilon j}}P\{\max_{0\leqslant i\leqslant\theta^j}\frac{Z(i,T^*_{kj})}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}\leqslant\frac{\theta}{1+\varepsilon}\}$$

$$+\sum_{|k|\leqslant\theta^{\varepsilon j}}P\{\max_{0\leqslant i\leqslant\theta^j}\sup_{T\in A_{kj}}\frac{|Z(i,T^*_{kj})-Z(i,T)|}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}\geqslant\frac{\theta\varepsilon}{(1+\varepsilon)^2}\}.$$

Noting that $Z(i,T)$ is an independent increment process for fixed $i$, we have

$$E(Z(i,T^*_{kj})-Z(i,T'_{kj}))^2=EZ^2(i,T^*_{kj})-EZ^2(i,T'_{kj})$$

$$=EY^2(i,T^*_{kj})-EY^2(i,T'_{kj})\leqslant\theta^{2(k+1)}-\theta^{2k}$$

$$\leqslant(\theta^2-1)H_1^2(\theta^{-j},T^*_{kj})$$

and hence for $1<\theta<1+\varepsilon^2/32$

$$\sum_{|k|\leqslant\theta^{\varepsilon j}}P\{\max_{0\leqslant i\leqslant\theta^j}\sup_{T\in A_{kj}}\frac{|Z(i,T^*_{kj})-Z(i,T)|}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}\geqslant\frac{\theta\varepsilon}{(1+\varepsilon)^2}\}\tag{3.5.51}$$

$$\leqslant\sum_{|k|\leqslant\theta^{\varepsilon j}}\sum_{i=0}^{\theta^j}P\{\sup_{T\in A_{kj}}\frac{|Z(i,T^*_{kj})-Z(i,T)|}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}\geqslant\frac{\varepsilon}{2}\}$$

$$\leqslant\sum_{|k|\leqslant\theta^{\varepsilon j}}\sum_{i=0}^{\theta^j}2P\{\frac{|Z(i,T^*_{kj})-Z(i,T'_{kj})|}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}\geqslant\frac{\varepsilon}{2}\}$$

$$\leqslant 4\sum_{|k|\leqslant\theta^{\varepsilon j}}\sum_{i=10}^{\theta^j}\exp\left(-\frac{\varepsilon^2\log\theta^{j+1}}{4(\theta^2-1)}\right)\leqslant 8\theta^{-2j}.$$

Using Condition (3.5.39) and the Slepian Lemma, we have

$$P\{\max_{0\leqslant i\leqslant\theta^j}\frac{Z(i,T^*_{kj})}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}\leqslant\frac{\theta}{1+\varepsilon}\}\tag{3.5.52}$$

$$\leqslant\prod_{i=0}^{[\theta^j]}P\{\frac{Z(i,T^*_{kj})}{H_1(\theta^{-j},T^*_{kj})(2\log\theta^{j+1})^{1/2}}\leqslant\frac{\theta}{1+\varepsilon}\}$$

$$\leqslant\prod_{i=0}^{[\theta^j]}\{1-\exp\left(-\frac{\theta^2}{1+\varepsilon}\log\theta^{j+1}\right)\}$$

$$\leqslant\exp(-\theta^{\varepsilon j/4})\leqslant\theta^{-2j}.$$

Therefore, we conclude from (3.5.50)—(3.5.52) that

$$P\left\{ \min_{|k| \leqslant \theta^{\varepsilon}} \inf_{T \in A_{kj}} \max_{0 \leqslant i \leqslant \theta^{j}} \frac{Y(i, T)}{H_1(\theta^{-j}, T)(2\log \theta^{j+1})^{1/2}} \leqslant \frac{1}{(1+\varepsilon)^2} \right\} \tag{3.5.53}$$

$$\leqslant 9\theta^{-2j}$$

for every $j$ large enough. Combining (3.5.47), (3.5.48), (3.5.53) with the Borel-Cantelli lemma yields

$$\varliminf_{T \to \infty} \sup_{|t| \leqslant 1} \frac{|X(t+a_T, T) - X(t, T)|}{H_1(a_T, T)(2\log(1/a_T))^{1/2}} \geqslant \frac{1}{(1+\varepsilon)^2} - \varepsilon \quad \text{a.s.} \tag{3.5.54}$$

This proves (3.5.46) by the arbitrariness of $\varepsilon$, and hence completes the proof of Theorem 3.5.1.

**Theorem 3.5.2** (Csörgő, Lin, Shao 1991) *Suppose that* (3.5.22), (3.5.23) *and* (3.5.24) *are satisfied with* $s_0 = a_T$, $u_0 = b_T$ *and that*

$$EX(R(js, s, ku, u))X(R(ms, s, lu, u)) \leqslant 0 \tag{3.5.55}$$

*for any* $s > 0$, $u > 0$, $j \neq k$ *and* $m \neq 1$. *Then*

$$\lim_{T \to \infty} \sup_{|t| \leqslant 1} \sup_{0 \leqslant v \leqslant 1} \frac{|X(R(t, a_T, v, b_T))|}{H_2(a_T, v, b_T)(2\log(1/a_T b_T))^{1/2}} = 1 \quad \text{a.s.} \tag{3.5.56}$$

$$\lim_{T \to \infty} \sup_{|t| \leqslant 1} \sup_{0 \leqslant v \leqslant 1} \sup_{0 \leqslant s \leqslant a_T} \sup_{0 \leqslant u \leqslant b_T} \frac{|X(R(t, s, v, u))|}{H_2(a_T, v, b_T)(2\log(1/a_T b_T))^{1/2}} = 1 \quad \text{a.s.} \tag{3.5.57}$$

*Proof* At first, we prove

$$\varlimsup_{T \to \infty} \sup_{|t| \leqslant 1} \sup_{0 \leqslant v \leqslant 1} \sup_{0 \leqslant s \leqslant a_T} \sup_{0 \leqslant u \leqslant b_T} \frac{|X(R(t, s, v, u))|}{H_2(a_T, v, b_T)(2\log(1/a_T b_T))^{1/2}}$$

$$\leqslant 1 \quad \text{a.s.} \tag{3.5.58}$$

Let $\theta > 1$. Define $A_{kj} = \{ T : \theta^{-(j+1)} < a_T \leqslant \theta^{-j}, \theta^{-(k+1)} < b_T \leqslant \theta^{-k} \}$, $j, k = 0, 1, \cdots$. We have

$$\varlimsup_{T \to \infty} \sup_{|t| \leqslant 1} \sup_{0 \leqslant v \leqslant 1} \sup_{0 \leqslant s \leqslant a_T} \sup_{0 \leqslant u \leqslant b_T} \frac{|X(R(t, s, v, u))|}{H_2(a_T, v, b_T)(2\log(1/a_T b_T))^{1/2}} \tag{3.5.59}$$

$$\leqslant \varlimsup_{j \to \infty} \sup_{k \geqslant 0} \sup_{T \in A_{kj}} \sup_{|t| \leqslant 1} \sup_{0 \leqslant v \leqslant 1} \sup_{0 \leqslant s \leqslant a_T} \sup_{0 \leqslant u \leqslant b_T}$$

$$\frac{|X(R(t,s,v,u))|}{H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2}}$$

$$\leqslant \overline{\lim_{j\to\infty}} \sup_{k\geqslant 0} \sup_{|t|\leqslant 1} \sup_{0\leqslant v\leqslant 1} \sup_{0\leqslant s\leqslant \theta^{-j}} \sup_{0\leqslant u\leqslant \theta^{-k}} \frac{(1+\varepsilon)|X(R(t,s,v,u))|}{H_2(\theta^{-j},v,\theta^{-k})(2\log\theta^{j+k})^{1/2}}$$

by Condition (3.5.24), provided that $\theta$ is close to one. Using Lemma 3.5.3, we derive

$$P\{\sup_{k\geqslant 0} \sup_{|t|\leqslant 1} \sup_{0\leqslant v\leqslant 1} \sup_{0\leqslant s\leqslant \theta^{-j}} \sup_{0\leqslant u\leqslant \theta^{-k}} \frac{|X(R(t,s,v,u))|}{H_2(\theta^{-j},v,\theta^{-k})(2\log\theta^{j+k})^{1/2}} \geqslant 1+\varepsilon\}$$

$$\leqslant C\sum_{k=0}^{\infty} \theta^{j+k}\exp\{-(1+\varepsilon)^2\log\theta^{j+k}\}$$

$$\leqslant C\theta^{-\varepsilon j}. \tag{3.5.60}$$

(3.5.58) follows from (3.5.59), (3,5,60) and the Borel-Cantelli lemma.

We now prove

$$\underline{\lim_{T\to\infty}} \sup_{|t|\leqslant 1} \sup_{0\leqslant v\leqslant 1} \frac{|X(R(t,a_T,v,b_T))|}{H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2}} \geqslant 1 \quad \text{a.s.} \tag{3.5.61}$$

Note that

$$\underline{\lim_{T\to\infty}} \sup_{|t|\leqslant 1} \sup_{0\leqslant v\leqslant 1} \frac{|X(R(t,a_T,v,b_T))|}{H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2}} \tag{3.5.62}$$

$$\geqslant \underline{\lim_{T\to\infty}} \inf_{k\geqslant 0} \inf_{T\in A_{kj}} \sup_{|t|\leqslant 1} \sup_{0\leqslant v\leqslant 1} \frac{|X(R(t,a_T,v,b_T))|}{H_2(a_T,v,b_T)(2\log(1/a_Tb_T))^{1/2}}$$

$$\geqslant \underline{\lim_{T\to\infty}} \inf_{k\geqslant 0} \sup_{0\leqslant t\leqslant 1} \sup_{0\leqslant v\leqslant 1} \frac{|X(R(t,\theta^{-j},v,\theta^{-k}))|}{H_2(\theta^{-j},v,\theta^{-k})(2\log\theta^{j+k+2})^{1/2}}$$

$$-\overline{\lim_{j\to\infty}} \sup_{k\geqslant 0} \sup_{0\leqslant t\leqslant 1} \sup_{\theta^{-k-1}\leqslant v\leqslant \theta^{-k-1}+1} \sup_{0\leqslant s\leqslant\theta^{-j}} \sup_{0\leqslant u\leqslant (1-\theta^{-1})\theta^{-k}}$$
$$\frac{(1+\varepsilon)|X(R(t,s,v,u))|}{H_2(\theta^{-j},v-\theta^{-k-1},\theta^{-k})(2\log\theta^{j+k})^{1/2}}$$

$$-\overline{\lim_{j\to\infty}} \sup_{k\geqslant 0} \sup_{\theta^{-j-1}\leqslant t\leqslant \theta^{-j-1}+1} \sup_{0\leqslant v\leqslant 1} \sup_{0\leqslant s\leqslant (1-\theta^{-1})\theta^{-j}} \sup_{0\leqslant u\leqslant \theta^{k}}$$
$$\frac{(1+\varepsilon)|X(R(t,s,v,u))|}{H_2(\theta^{-j},v,\theta^{-k})(2\log\theta^{j+k})^{1/2}}$$

$$\geqslant \underline{\lim_{j\to\infty}} \inf_{k\geqslant 0} \max_{0\leqslant l\leqslant \theta^{j}} \max_{0\leqslant m\leqslant \theta^{k}} \frac{|X(R(l\theta^{-j},\theta^{-j},m\theta^{-k},\theta^{-k}))|}{H_2(\theta^{-j},m\theta^{-k},\theta^{-k})(2\log\theta^{j+k+2})^{1/2}}$$

$$-\varepsilon \varlimsup_{j\to\infty} \sup_{k\geq 0} \sup_{0\leq t\leq 1} \sup_{0\leq v\leq \theta^{k-1}+1} \sup_{0\leq s\leq \theta^{-j}} \sup_{0\leq u\leq (1-\theta^{-1})\theta^{-k}} \frac{(1+\varepsilon)|X(R(t,s,v,u))|}{H_2(\theta^{-j},v,(1-\theta^{-1})\theta^{-k})(2\log \theta^{j+k})^{1/2}}$$

$$-\varepsilon \varlimsup_{j\to\infty} \sup_{k\geq 0} \sup_{0\leq t\leq \theta^{j-1}+1} \sup_{0\leq v\leq 1} \sup_{0\leq s\leq (1-\theta^{-1})\theta^{-j}} \sup_{0\leq u\leq \theta^{-k}} \frac{(1+\varepsilon)|X(R(t,s,v,u))|}{H_2((1-\theta^{-1})\theta^{-j},v,\theta^{-k})(2\log \theta^{j+k})^{1/2}}$$

$$=: I_1 - I_2 - I_3,$$

where Condition (3.5.24) is used again. Along the lines of the proof of (3.5.58) and recalling Remark 3.5.1, we can obtain

$$I_2 + I_3 \leq 3\varepsilon \qquad \text{a.s.} \tag{3.5.63}$$

For $I_1$, in terms of (3.5.55), we can apply the Slepian Lemma and get

$$P\left\{ \inf_{k\geq 0} \max_{0\leq l\leq \theta^j} \max_{0\leq m\leq \theta^k} \frac{|X(R(l\theta^{-j},\theta^{-j},m\theta^{-k},\theta^{-k}))|}{H_2(\theta^{-j},m\theta^{-k},\theta^{-k})(2\log \theta^{j+k+2})^{1/2}} \leq 1-\varepsilon \right\}$$

$$\leq \sum_{k=0}^{\infty} \prod_{l=0}^{[\theta^j]} \prod_{m=0}^{[\theta^k]} P\left\{ \frac{|X(R(l\theta^{-j},\theta^{-j},m\theta^{-k},\theta^{-k}))|}{H_2(\theta^{-j},m\theta^{-k},\theta^{-k})(2\log\theta^{j+k+2})^{1/2}} \leq 1-\varepsilon \right\}$$

$$\leq \sum_{k=0}^{\infty} \{1-\exp(-(1-\varepsilon)\log \theta^{j+k+2})\}^{\theta^{j+k}}$$

$$\leq \sum_{k=0}^{\infty} \exp(-\theta^{\varepsilon(j+k)}) \leq c\exp(-\theta^{\varepsilon j})$$

which implies that

$$I_1 \geq 1-\varepsilon \qquad \text{a.s.} \tag{3.5.64}$$

Combining (3.5.63), (3.5.64), with (3.5.62) yields (3.5.61). This completes the proof of Theorem 3.5.2.

The following corollaries deal with the examples given above.

**Corollary 3.5.1** *Let $\{W(x,y);\ -\infty<x<\infty,\ 0\leq y<\infty\}$ be a standard two-parameter Wiener process. Then*

$$\lim_{T\to\infty} \sup_{|t|\leq 1} \sup_{0\leq v\leq 1} \frac{|W(R(t,a_T,v,b_T))|}{(2a_Tb_T\log(1/a_Tb_T))^{1/2}} = 1 \qquad \text{a.s.}$$

$$\lim_{T\to\infty}\sup_{|t|\leqslant 1}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\sup_{0\leqslant u\leqslant b_T}\frac{|W(R(t,s,v,u))|}{(2a_Tb_T\log(1/a_Tb_T))^{1/2}}=1\quad \text{a.s.}$$

*If, in addition,*

$$\log\log(Ta_T+(Ta_T)^{-1})=o(\log(1/a_T))\quad \text{as}\quad T\to\infty, \tag{3.5.65}$$

*then*

$$\lim_{T\to\infty}\sup_{|t|<1}\frac{|W(t+a_T,T)-W(t,T)|}{(2Ta_T\log(1/a_T))^{1/2}}=1\quad \text{a.s.}$$

$$\lim_{T\to\infty}\sup_{|t|\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\frac{|W(t+s,T)-W(t,T)|}{(2Ta_T\log(1/a_T))^{1/2}}=1\quad \text{a.s.}$$

(cf. Theorems 1.14.2 and S1.14.2 of Csörgő and Révész 1981).

**Corollary 3.5.2** *Let* $\{K(x,y);\ 0\leqslant x\leqslant 1, 0\leqslant y<\infty\}$ *be a Kiefer process. Then*

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant 1-a_T}\sup_{0\leqslant v\leqslant 1}\frac{|K(R(t,a_T,v,b_T))|}{(2a_T(1-a_T)b_T\log(1/a_Tb_T))^{1/2}}=1\quad \text{a.s.}$$

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant 1-a_T}\sup_{0\leqslant v\leqslant 1}\sup_{0\leqslant s\leqslant a_T}\sup_{0\leqslant u\leqslant b_T}\frac{|K(R(t,s,v,u))|}{(2a_T(1-a_T)b_T\log(1/a_Tb_T))^{1/2}}=1\quad \text{a.s.}$$

*If, in addition* (3.5.65) *is satisfied, then*

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant 1-a_T}\frac{|K(t+a_T,T)-K(t,T)|}{(2Ta_T(1-a_T)\log(1/a_T))^{1/2}}=1\quad \text{a.s.}$$

$$\lim_{T\to\infty}\sup_{0\leqslant t\leqslant 1-a_T}\sup_{0\leqslant s\leqslant a_T}\frac{|K(t+s,T)-K(t,T)|}{(2Ta_T(1-a_T)\log(1/a_T))^{1/2}}=1\quad \text{a.s.}$$

**Corollary 3. 5. 3** *Let* $\{X(t,v,);\ -\infty<t<\infty,\ 0\leqslant v<\infty\}$ *be a two-parameter Ornstein-Uhlenbeck process as in Example* 3. *Suppose that there exists* $c_0>0$ *such that*

$$\int_{0<x\leqslant T,\lambda(x)\geqslant 1/s}\frac{\gamma(x)}{\lambda(x)}\,dx\leqslant c_0 s\int_{0\leqslant x\leqslant T,\lambda(x)\leqslant 1/s}\gamma(x)dx$$

*for any* $0<s\leqslant a_T$, *and that*

$$\log\log(H_1(a_T,T)+H_1^{-1}(a_T,T))=o(\log(1/a_T))\text{ as }T\to\infty,$$

*where* $H_1^2(a_T, T) = 2\int_0^T \frac{\gamma(x)}{\lambda(x)}(1 - \exp(-\lambda(x)a_T))dx$. *Then*

$$\lim_{T\to\infty} \sup_{0\le t\le 1} \frac{|X(t+a_T, T) - X(t,T)|}{H_1(a_T, T)(2\log(1/a_T))^{1/2}} = 1 \quad \text{a.s.}$$

$$\lim_{T\to\infty} \sup_{0\le t\le 1} \sup_{0\le s\le a_T} \frac{|X(t+s,T) - X(t,T)|}{H_1(a_T,T)(2\log(1/a_T))^{1/2}} = 1 \quad \text{a.s.}$$

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